

C^1 -CLASSIFICATION OF GAPPED PARENT HAMILTONIANS OF QUANTUM SPIN CHAINS

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ABSTRACT. We consider the C^1 -classification of gapped Hamiltonians introduced in [FNW, N] as parents Hamiltonians of translation invariant finitely correlated states. Within this family, we show that the number of edge modes, which is equal at the left and right edge, is the complete invariant. The construction proves that translation invariance of the ‘bulk’ ground state does not need to be broken to establish C^1 -equivalence, namely that the spin chain does not need to be blocked.

1. INTRODUCTION

A phase transition refers to a qualitative change in the properties of a family of physical systems as a parameter crosses a critical value, as for example the breaking of a continuous symmetry in thermal states as the temperature changes. The term *quantum phase transition* [S] refers somewhat unluckily to transitions happening at zero temperature, in particular qualitative changes in the ground states of quantum systems depending on a parameter. The archetypal example is here the transition from a unique ground state to a two-dimensional ground state space happening in the Ising model in a transverse magnetic field, which is accompanied by the closing of the spectral gap above the ground state energy. Such ground state phases and the transitions between them have received renewed attention recently both for fundamental reasons and for their potential applications in quantum information theory, with a particular focus on the structure of entanglement in the ground states. It is a natural and important question to consider the classification of gapped Hamiltonians, namely of Hamiltonians that have uniform spectral gap above the ground state energy [HW, CGW1, CGW2].

In the context of quantum spin systems, a widely accepted criterion for the classification of gapped Hamiltonians is as follows: two gapped Hamiltonians are equivalent if and only if they are connected by a continuous path of uniformly gapped Hamiltonians [CGW1, CGW2]. In this article, we consider a bit stronger version of this, a C^1 -equivalence. We say two gapped Hamiltonians are C^1 -equivalent if and only if they are connected by a continuous and piecewise C^1 -path of uniformly gapped Hamiltonians. We call the classification of gapped Hamiltonians with respect to this equivalence relation, the C^1 -classification of gapped Hamiltonians.

In [BMNS], it was shown that the ‘ground state structure’ is an invariant of this C^1 -classification of gapped Hamiltonians. The statement of [BMNS] is quite general and in particular does not refer to the spatial dimension of the spin system. The need to prove the existence of a uniform spectral gap however makes the construction of relevant examples a hard problem, in particular in higher dimensions. In one dimension, the martingale method has been successfully applied to a large class of models, namely to systems with frustration free, finitely correlated ground states [FNW, N]. They are simple, yet correlated states, and [FNW] gives a general recipe to construct gapped Hamiltonians which have a finitely correlated ground states, with a simple control of the spectral gap above the ground state energy.

For translation-invariant one dimensional models, we consider the three possible infinite volume limits of finite volume ground states: on the bi-infinite chain – the *bulk ground states* – and on the two possible half-infinite chains – the *left/right edge states*. By [BMNS], the dimensions of these three ground state spaces are invariants of the C^1 -classification. Of particular interest is the case of a unique, translation invariant bulk state, where the index is reduced to the pair of the numbers of edge modes. It is in general not clear if it is the *complete invariant*, namely if it uniquely determines the phase of the gapped Hamiltonians in the class. The Hamiltonians in [FNW] have symmetric edge states, namely an equal number left and right edge modes, and the spin-1 antiferromagnetic model introduced by Affleck-Lieb-Kennedy-Tasaki in [AKLT] is a physically relevant element of that class. On the other hand, the gapped models introduced in [BN1, BN2], the ‘PVBS

models', which also have a unique finitely correlated ground state in the bulk, have asymmetric edge modes and these Hamiltonians do not belong to the class given in [FNW].

The recurrent claim that all such one-dimensional frustration free models belong to the same phase and in particular that they are all equivalent to a pure product state, see e.g. [SPC], was refined and partly clarified in [BN1, BN2] for PVBS models (but see [WOVC] for explicit constructions of quantum phase transitions in families of finitely correlated states). There, it was shown that for the restricted class of PVBS models, the dimensions of edge states on the two possible half-infinite chains are indeed complete invariant. It is further shown that the AKLT model belongs to one of these phases, namely the phase of a PVBS model with two-dimensional ground state space at each edge.

In this paper, we prove that the number of edge modes is indeed the complete invariant of the C^1 -classification within the family of gapped Hamiltonians of [FNW]. Explicitly, we construct a smooth path of uniformly gapped Hamiltonians between any two given elements of the family which have the same number of edge modes. Importantly, and unlike [SPC], we do so *without blocking sites*, and hence without breaking translation invariance of the ground state to a finite number of periodic states. Moreover, the interaction range along the path can be chosen to be constant and we give an explicit upper bound on the shortest such range.

Notations. We denote the Euclidean distance between a point x and a subset M in \mathbb{R}^k by $d_{\mathbb{R}^k}(x, M)$. We also denote the Euclidean distance between two subsets M_1, M_2 in \mathbb{R}^k by $d_{\mathbb{R}^k}(M_1, M_2)$. Similarly, we denote the Euclidean distance between a point x and a subset S in \mathbb{C} (resp. \mathbb{R}) by $d_{\mathbb{C}}(x, S)$ (resp. $d_{\mathbb{R}}(x, S)$). For a subset S of \mathbb{C} and $\delta > 0$, the δ -neighborhood of S is denoted by S_δ . We denote the open ball in \mathbb{C} centered at $x \in \mathbb{C}$ with radius r by $B_r(x)$. For a linear operator T , we denote the spectrum of T by $\sigma(T)$, and the spectral radius of T by r_T . For an isolated subset S of $\sigma(T)$, we denote the spectral projection of T onto S by P_S^T . If T is self-adjoint and S is a subset of \mathbb{R} , then $\text{Proj}[T \in S]$ also indicates the spectral projection of T corresponding to $\sigma(T) \cap S$. For $k \in \mathbb{N}$, the set of orthogonal projections in $k \times k$ matrices $\text{Mat}_k(\mathbb{C})$ is denoted by $\mathcal{P}(\text{Mat}_k(\mathbb{C}))$ and the set of positive elements of $\text{Mat}_k(\mathbb{C})$ by $\text{Mat}_k(\mathbb{C})_+$. We write $A > 0$ for $A \in \text{Mat}_k(\mathbb{C})$ if A is strictly positive. For $k \in \mathbb{N}$, $\text{Tr}_{\text{Mat}_k(\mathbb{C})}$ denotes the trace on $\text{Mat}_k(\mathbb{C})$. For a finite dimensional Hilbert space, bracket \langle, \rangle denotes the inner product of the space under consideration. For a Hilbert space \mathfrak{H} , we denote the set of all bounded linear operators on \mathfrak{H} by $B(\mathfrak{H})$.

2. THE SETUP AND MAIN RESULT

For $\mathbb{N} \ni n \geq 2$, let \mathcal{A} be the finite dimensional C^* -algebra $\mathcal{A} = \text{Mat}_n(\mathbb{C})$, the algebra of $n \times n$ matrices. Throughout this article, this n is fixed as the dimension of the spin under consideration. We denote the set of all finite subsets in $\Gamma \subset \mathbb{Z}$ by \mathfrak{S}_Γ . The number of elements in a finite set $\Lambda \subset \mathbb{Z}$ is denoted by $|\Lambda|$. When we talk about intervals in \mathbb{Z} , $[a, b]$ for $a \leq b$, means the interval in \mathbb{Z} , i.e., $[a, b] \cap \mathbb{Z}$. We denote the set of all finite intervals in Γ by \mathfrak{I}_Γ . For each $z \in \mathbb{Z}$, we let $\mathcal{A}_{\{z\}}$ be an isomorphic copy of \mathcal{A} and for any finite subset $\Lambda \subset \mathbb{Z}$, $\mathcal{A}_\Lambda = \otimes_{z \in \Lambda} \mathcal{A}_{\{z\}}$ is the local algebra of observables. For finite Λ , the algebra \mathcal{A}_Λ can be regarded as the set of all bounded operators acting on a Hilbert space $\otimes_{z \in \Lambda} \mathbb{C}^n$. We use this identification freely, and we denote the trace of $\mathcal{A}_\Lambda \simeq B(\otimes_{z \in \Lambda} \mathbb{C}^n)$ by Tr_Λ . Throughout this article, we fix an orthonormal basis $\{\psi_\mu\}_{\mu=1}^n$ of \mathbb{C}^n . If $\Lambda_1 \subset \Lambda_2$, the algebra \mathcal{A}_{Λ_1} is naturally embedded in \mathcal{A}_{Λ_2} by tensoring its elements with the identity. Finally, for an infinite subset Γ of \mathbb{Z} , the algebra \mathcal{A}_Γ is given as the inductive limit of the algebras \mathcal{A}_Λ with $\Lambda \in \mathfrak{S}_\Gamma$. In particular, $\mathcal{A}_{\mathbb{Z}}$ is the chain algebra. We denote the set of local observables in Γ by $\mathcal{A}_\Gamma^{\text{loc}} = \bigcup_{\Lambda \in \mathfrak{S}_\Gamma} \mathcal{A}_\Lambda$.

For any $x \in \mathbb{Z}$, let τ_x be the shift operator by x on $\mathcal{A}_{\mathbb{Z}}$. An interaction is a map Φ from $\mathfrak{S}_{\mathbb{Z}}$ into $\mathcal{A}_{\mathbb{Z}}^{\text{loc}}$ such that $\Phi(X) \in \mathcal{A}_X$ and $\Phi(X) = \Phi(X)^*$ for $X \in \mathfrak{S}_{\mathbb{Z}}$. An interaction Φ is translation invariant if $\Phi(X + j) = \tau_j(\Phi(X))$, for all $j \in \mathbb{Z}$ and $X \in \mathfrak{S}_{\mathbb{Z}}$. Furthermore, it is of finite range if there exists an $m \in \mathbb{N}$ such that $\Phi(X) = 0$, for X with diameter larger than m . In this case, we say that the interaction length of Φ is less than or equal to m . A Hamiltonian associated with Φ is a net of self-adjoint operators $H := (H_\Lambda)_{\Lambda \in \mathfrak{I}_{\mathbb{Z}}}$ such that

$$(1) \quad H_\Lambda := \sum_{X \subset \Lambda} \Phi(X).$$

Note that $H_\Lambda \in \mathcal{A}_\Lambda$. Without loss of generality we consider positive interactions i.e., $\Phi(X) \geq 0$ for any $X \in \mathfrak{S}_{\mathbb{Z}}$, throughout this article. We denote the set of all positive translation invariant finite range interactions

by \mathcal{J} . Furthermore, for $m \in \mathbb{N}$, we denote by \mathcal{J}_m the set of all positive translation invariant interactions with interaction length less than or equal to m .

For a finite interval Λ , a ground state of H_Λ means a state on \mathcal{A}_Λ with support in the lowest eigenvalue space of H_Λ . We denote the set of all ground states of H_Λ on \mathcal{A}_Λ by $\mathcal{S}_\Lambda(H)$. For $\Lambda \in \mathcal{I}_\Gamma$, any of the elements in $\mathcal{S}_\Lambda(H)$ can be extended to a state on \mathcal{A}_Γ , and there exists a weak-* accumulation points of such extensions, in the thermodynamical limit $\Lambda \rightarrow \Gamma$. We denote the set of all such accumulation points by $\mathcal{S}_\Gamma(H)$.

Let us specify what we mean by a *gapped* Hamiltonian:

Definition 2.1. A Hamiltonian $H := (H_\Lambda)_{\Lambda \in \mathcal{I}_\mathbb{Z}}$ associated with a positive translation invariant finite range interaction is *gapped* if there exists $\gamma > 0$ and $N_0 \in \mathbb{N}$ such that the difference between the smallest and the next-smallest eigenvalue of H_Λ , is bounded below by γ , for all finite intervals $\Lambda \subset \mathbb{Z}$ with $|\Lambda| \geq N_0$.

We call this γ a gap of the Hamiltonian H . Note that the lowest eigenvalue may be degenerate.

Gapped ground state phases. Now we introduce the C^1 -classification of gapped Hamiltonians. We say $\Phi : [0, 1] \ni t \mapsto \Phi(t) \in \mathcal{J}$ is a continuous and piecewise C^1 -path if for each $X \in \mathfrak{S}_\mathbb{Z}$, $[0, 1] \ni t \mapsto \Phi(t; X) \in \mathcal{A}_X$ is continuous and piecewise C^1 with respect to the norm topology.

Definition 2.2 (C^1 -classification of gapped Hamiltonians). Let H_0, H_1 be gapped Hamiltonians associated with interactions $\Phi_0, \Phi_1 \in \mathcal{J}$. We say that H_0, H_1 are C^1 -equivalent if the following conditions are satisfied.

- (i) There exists $m \in \mathbb{N}$ and a continuous and piecewise C^1 -path $\Phi : [0, 1] \rightarrow \mathcal{J}_m$ such that $\Phi(0) = \Phi_0$, $\Phi(1) = \Phi_1$.
- (ii) Let $H(t)$ be the Hamiltonian associated with $\Phi(t)$ for each $t \in [0, 1]$. There are $\gamma > 0$, $N_0 \in \mathbb{N}$, and finite intervals $I(t) = [a(t), b(t)]$, whose endpoints $a(t), b(t)$ smoothly depending on $t \in [0, 1]$, such that for all finite intervals $\Lambda \subset \mathbb{Z}$ with $|\Lambda| \geq N_0$, the smallest eigenvalue of $H(t)_\Lambda$ is in $I(t)$ and the rest of the spectrum is in $[b(t) + \gamma, \infty)$.

The advantage of considering C^1 -paths over just continuous ones is as follows. The following theorem is a special case of the Theorem 5.5 of [BMNS].

Theorem 2.3 ([BMNS]). Suppose that two gapped Hamiltonians H_0, H_1 are C^1 -equivalent. Then, for $\Gamma = (-\infty, -1] \cap \mathbb{Z}$, $\Gamma = [0, \infty) \cap \mathbb{Z}$ and $\Gamma = \mathbb{Z}$, there exists a quasi-local automorphism α_Γ of \mathcal{A}_Γ such that

$$\mathcal{S}_\Gamma(H_1) = \mathcal{S}_\Gamma(H_0) \circ \alpha_\Gamma.$$

See [BMNS] for the general statement and the definition of quasi-locality. In other words, the structure of the bulk and of the left/right edge ground state spaces are invariants of the C^1 -classification.

Intersection property and parent Hamiltonians. Let \mathcal{D}_N be a subspace of $\otimes_{i=0}^{N-1} \mathbb{C}^n$, for each $N \in \mathbb{N}$. We say that the sequence of subspaces $\{\mathcal{D}_N\}_{N \in \mathbb{N}}$ satisfies the *intersection property*, if there exists an $m \in \mathbb{N}$, such that the relation

$$(2) \quad \mathcal{D}_N = \bigcap_{x=0}^{N-m} (\mathbb{C}^n)^{\otimes x} \otimes \mathcal{D}_m \otimes (\mathbb{C}^n)^{\otimes N-m-x},$$

holds for all $N \geq m$. In order to specify the number $m \in \mathbb{N}$, we will say that $\{\mathcal{D}_N\}_{N \in \mathbb{N}}$ satisfies Property (I, m) when (2) holds for m and all $N \geq m$. Note that Property (I, m) implies Property (I, m') for all $m' \geq m$.

Given a sequence of nonzero spaces $\{\mathcal{D}_N\}$ satisfying Property (I, m), there is a natural positive interaction for which \mathcal{D}_N are the ground state spaces of the corresponding Hamiltonian. Namely let Q_m be the orthogonal projection onto the orthogonal complement of \mathcal{D}_m in $\otimes_{i=0}^{m-1} \mathbb{C}^n$, and define

$$\Phi(X) := \begin{cases} \tau_x(Q_m), & \text{if } X = [x, x+m-1] \text{ for some } x \in \mathbb{Z} \\ 0, & \text{otherwise} \end{cases}.$$

By (2), we see that $\ker H_{[0, N-1]} = \mathcal{D}_N$, for $N \geq m$, for the Hamiltonian $H = (H_\Lambda)$ associated with Φ . We shall refer to that particular Hamiltonian as the *parent Hamiltonian* of $\{\mathcal{D}_N\}$.

Gapped Hamiltonians in [FNW]. Now we recall the class of Hamiltonians introduced in [FNW]. For $k \in \mathbb{N}$, let \mathcal{T}_k be the set of all primitive completely positive maps on $\text{Mat}_k(\mathbb{C})$ with spectral radius 1. It is well

known that for each $T \in \mathcal{T}_k$, $e_T := P_{\{1\}}^T(1)$ is positive and invertible. Furthermore, there exists a faithful state φ_T given by $P_{\{1\}}^T(a) = \varphi_T(a)e_T$ for $a \in \text{Mat}_k(\mathbb{C})$. There exists a positive invertible element ρ_T in $\text{Mat}_k(\mathbb{C})$ with $\varphi_T = \text{Tr}_{\text{Mat}_k(\mathbb{C})}(\rho_T \cdot)$. Note that φ_T is T -invariant and $\varphi_T(e_T) = 1$ (see Appendix D). For $T \in \mathcal{T}_k$, we denote $a_T := \|e_T^{-1}\|$ and $c_T := \|\rho_T^{-1}\|$. Clearly, $0 < a_T, c_T < \infty$.

For each n -tuple of $k \times k$ -matrices $\mathbb{B} = (B_1, \dots, B_n) \in \text{Mat}_{1,n}(\text{Mat}_k(\mathbb{C}))$, we define a completely positive map $\widehat{\mathbb{E}}^{\mathbb{B}}$ on $\text{Mat}_k(\mathbb{C})$ by

$$\widehat{\mathbb{E}}^{\mathbb{B}} := \sum_{\mu=1}^n B_{\mu} \cdot B_{\mu}^*,$$

and with this,

$$B_{n,k} := \left\{ \mathbb{B} = (B_1, \dots, B_n) \in \text{Mat}_{1,n}(\text{Mat}_k(\mathbb{C})) \mid \widehat{\mathbb{E}}^{\mathbb{B}} \in \mathcal{T}_k \right\}.$$

Now, for $N \in \mathbb{N}$ and given $\mathbb{B} \in \text{Mat}_{1,n}(\text{Mat}_k(\mathbb{C}))$, define $\Gamma_N^{k,\mathbb{B}} : \text{Mat}_k(\mathbb{C}) \rightarrow \otimes_{i=0}^{N-1} \mathbb{C}^n$ by

$$(3) \quad \Gamma_N^{k,\mathbb{B}}(C) := \sum_{\mu_1, \dots, \mu_N=1}^n \text{Tr}_{\text{Mat}_k(\mathbb{C})}(C B_{\mu_N}^* \cdots B_{\mu_1}^*) \psi_{\mu_1} \otimes \cdots \otimes \psi_{\mu_N}, \quad C \in \text{Mat}_k(\mathbb{C}).$$

Furthermore, set $\mathcal{G}_N^{k,\mathbb{B}} := \text{Ran } \Gamma_N^{k,\mathbb{B}}$, and let $G_N^{k,\mathbb{B}}$ be the orthogonal projection onto $\mathcal{G}_N^{k,\mathbb{B}}$ in $\otimes_{i=0}^{N-1} \mathbb{C}^n$. This gives us a sequence of subspaces $\{\mathcal{G}_N^{k,\mathbb{B}}\}_N$, and we shall say that (k, \mathbb{B}) satisfies Property (I, m) if the spaces $\{\mathcal{G}_N^{k,\mathbb{B}}\}$ do so. For (k, \mathbb{B}) , we set

$$m^{k,\mathbb{B}} := \min \{m \in \mathbb{N} \mid (k, \mathbb{B}) \text{ satisfies Property (I, } m)\}.$$

Let \mathfrak{E}_d denote the set of states on $\text{Mat}_d(\mathbb{C})$ for $d \in \mathbb{N}$.

Proposition 2.4. *Let $k \in \mathbb{N}$, and $\mathbb{B} \in B_{n,k}$.*

- (i) *(k, \mathbb{B}) satisfies the intersection property.*
- (ii) *For each $m \in \mathbb{N}$, let $\Phi_m^{k,\mathbb{B}}$ be a positive interaction given by*

$$(4) \quad \Phi_m^{k,\mathbb{B}}(X) := \begin{cases} \tau_x(1 - G_m^{k,\mathbb{B}}), & \text{if } X = [x, x+m-1] \text{ for some } x \in \mathbb{Z}, \\ 0, & \text{otherwise} \end{cases},$$

and $H_m^{k,\mathbb{B}}$ the Hamiltonian associated with $\Phi_m^{k,\mathbb{B}}$. Then for any $m \geq m^{k,\mathbb{B}}$,

- (a) *$H_m^{k,\mathbb{B}}$ is gapped,*
- (b) *$\mathcal{S}_{\mathbb{Z}}(H_m^{k,\mathbb{B}})$ consists of a unique state $\omega_{\infty}^{\mathbb{B}}$ on $\mathcal{A}_{\mathbb{Z}}$,*
- (c) *there exist affine bijections*

$$\Xi_L : \mathfrak{E}_k \rightarrow \mathcal{S}_{(-\infty, -1]}(H_m^{k,\mathbb{B}}), \quad \Xi_R : \mathfrak{E}_k \rightarrow \mathcal{S}_{[0, +\infty)}(H_m^{k,\mathbb{B}}).$$

Let \mathcal{H} denote the set of all gapped Hamiltonians given by the recipe of above proposition. We note that \mathcal{H} corresponds to the Hamiltonians introduced in [FNW].

Classification of the parent Hamiltonians. We now introduce a class of Hamiltonians labeled by $k \in \mathbb{N}$ as

$$\mathcal{H}_k := \{H_m^{k,\mathbb{B}} \mid \mathbb{B} \in B_{n,k}, \mathbb{N} \ni m \geq m^{k,\mathbb{B}}\}.$$

By the definition, we have

$$\mathcal{H} = \cup_{k \in \mathbb{N}} \mathcal{H}_k,$$

and from Proposition 2.4(iii), \mathcal{H}_k is the class of Hamiltonians in \mathcal{H} whose left and right edge state spaces are both of dimension k .

The following theorem which is the main result of this paper is that this dimension k is the complete invariant for the C^1 -classification within the set \mathcal{H} :

Theorem 2.5. *For $k, k' \in \mathbb{N}$, let H, H' be gapped Hamiltonians on $\mathcal{A}_{\mathbb{Z}}$ such that $H \in \mathcal{H}_k$ and $H' \in \mathcal{H}_{k'}$. Then H and H' are C^1 -equivalent gapped Hamiltonians if and only if $k = k'$. Furthermore, the path of interactions in Definition 2.2(i) can be taken to be in \mathcal{J}_m , for $m = \max\{M, M'\}$, where $M, M' \in \mathbb{N}$ are numbers which depend only on H, H' respectively.*

A classification of these Hamiltonians in [FNW, N] is considered in [SPC], but there, the path is allowed to take *periodic* interactions, instead of *translation invariant* interactions. Allowing periodic interactions corresponds to considering the situation $n \geq k^2$, which is much easier. To find a path with uniform gap connecting H_m^{k, \mathbb{B}_0} and H_m^{k, \mathbb{B}_1} , we need to find a path of $\widehat{\mathbb{E}}^{\mathbb{B}(t)}$ satisfying $\mathbb{B}(0) = \mathbb{B}_0$, $\mathbb{B}(1) = \mathbb{B}_1$. By the definition, these completely positive maps $\widehat{\mathbb{E}}^{\mathbb{B}(t)}$ should have Kraus rank less than or equal to n , and the gap remains open if they are primitive. But if $n \geq k^2$, the Kraus rank is less than or equal to n for any kind of completely positive map on $\text{Mat}_k(\mathbb{C})$. Therefore, we may just take a path $(1-t)\widehat{\mathbb{E}}^{\mathbb{B}_0} + t\widehat{\mathbb{E}}^{\mathbb{B}_1}$, which is primitive because of the primitivity of $\widehat{\mathbb{E}}^{\mathbb{B}_0}$ and $\widehat{\mathbb{E}}^{\mathbb{B}_1}$, and with Kraus rank less than or equal to $k^2 \leq n$. Translation invariance requires an additional work, because we can not take such a simple path: Section 5 is devoted to this problem, covering the case $k^2 < n$.

This paper is organized as follows. In Section 3, we summarize the ground state structure of Hamiltonians $H_m^{k, \mathbb{B}}$. The classification Theorem 2.5 is proven in Section 4. The most nontrivial ingredient of the proof is the deformation of \mathbb{B} , namely the pathwise connectedness of $B_{n, k}$, which we shall prove in Section 5.

3. THE GROUND STATE STRUCTURE OF $H_m^{k, \mathbb{B}}$

In this section, we recall the spectral property of $H_m^{k, \mathbb{B}}$ and characterize the edge ground states on left and right half-infinite chains. For $k \in \mathbb{N}$ and $\mathbb{B} \in B_{n, k}$, we shall denote $e_{\mathbb{B}} = e_{\widehat{\mathbb{E}}^{\mathbb{B}}}$, $\varphi^{\mathbb{B}} = \varphi_{\widehat{\mathbb{E}}^{\mathbb{B}}}$, and $a^{\mathbb{B}} = a_{\widehat{\mathbb{E}}^{\mathbb{B}}}$, $c^{\mathbb{B}} = c_{\widehat{\mathbb{E}}^{\mathbb{B}}}$, $r_{\mathbb{B}} = r_{\widehat{\mathbb{E}}^{\mathbb{B}}}$.

3.1. The parent Hamiltonian and its spectral gap. We first recall the results proven in [FNW]. Let $k \in \mathbb{N}$ and $\mathbb{B} \in B_{n, k}$. For each $N \in \mathbb{N}$, let

$$\begin{aligned} E^{k, \mathbb{B}}(N) &:= k a^{\mathbb{B}} c^{\mathbb{B}} \left\| (\widehat{\mathbb{E}}^{\mathbb{B}})^N (1 - P_{\{1\}}^{\mathbb{B}}) \right\|. \\ F^{k, \mathbb{B}} &:= \frac{4}{a^{\mathbb{B}} c^{\mathbb{B}}} \left(\sup_N E^{k, \mathbb{B}}(N) + c^{\mathbb{B}} + a^{\mathbb{B}} \text{Tr}_{\text{Mat}_k(\mathbb{C})} e_{\mathbb{B}} \right), \\ L^{k, \mathbb{B}} &:= \min \left\{ L \in \mathbb{N} : \sup_{N \geq L} E^{k, \mathbb{B}}(N) < \frac{1}{2} \right\}. \\ \bar{l}^{k, \mathbb{B}} &:= \min \left\{ l \in \mathbb{N} : \sup_{N \geq l} \left(\sqrt{N+1} (3E^{k, \mathbb{B}}(N) F^{k, \mathbb{B}} + 2) (E^{k, \mathbb{B}}(N) F^{k, \mathbb{B}}) + E^{k, \mathbb{B}}(N) \right) < 1 \right\}. \end{aligned}$$

As $\widehat{\mathbb{E}}^{\mathbb{B}} \in \mathcal{T}_k$, its spectrum satisfies $\sigma(\widehat{\mathbb{E}}^{\mathbb{B}}) \setminus \{1\} \subset \{z \in \mathbb{C} \mid |z| < 1\}$. Therefore, $E^{k, \mathbb{B}}(N)$ goes to 0 exponentially fast with respect to N . It follows that $F^{k, \mathbb{B}}$, $L^{k, \mathbb{B}}$, $\bar{l}^{k, \mathbb{B}}$ are well-defined and finite. By the definition, clearly, $L^{k, \mathbb{B}} \leq \bar{l}^{k, \mathbb{B}}$.

Proposition 3.1. *Let $k \in \mathbb{N}$, and $\mathbb{B} \in B_{n, k}$. For each $m \in \mathbb{N}$, let $\Phi_m^{k, \mathbb{B}}$ be a positive interaction given by*

$$(5) \quad \Phi_m^{k, \mathbb{B}}(X) := \begin{cases} \tau_x (1 - G_m^{k, \mathbb{B}}), & \text{if } X = [x, x + m - 1] \text{ for some } x \in \mathbb{Z} \\ 0, & \text{otherwise} \end{cases},$$

and $H_m^{k, \mathbb{B}}$ be the translation invariant Hamiltonian associated with $\Phi_m^{k, \mathbb{B}}$. Then for any $N, m \in \mathbb{N}$ with $N \geq m \geq m^{k, \mathbb{B}}$, the lowest eigenvalue of $(H_m^{k, \mathbb{B}})_{[0, N-1]}$ is 0 and the corresponding spectral projection is $G_N^{k, \mathbb{B}}$. Furthermore, for all $l \in \mathbb{N}$ with $\max\{\bar{l}^{k, \mathbb{B}}, m\} < l < N$,

$$\frac{\gamma_{l, m}^{k, \mathbb{B}}}{4(l+2)} (1 - G_N^{k, \mathbb{B}}) \leq (H_m^{k, \mathbb{B}})_{[0, N-1]}.$$

Here,

$$\gamma_{l, m}^{k, \mathbb{B}} = d_{\mathbb{R}} \left(\sigma \left((H_m^{k, \mathbb{B}})_{[0, l-1]} \right) \setminus \{0\}, 0 \right)$$

is the spectral gap of the finite volume Hamiltonian on an interval of length l .

We refer to [FNW] for the proof of the proposition. One crucial ingredient is the proof of the Intersection Property for the sets $\mathcal{G}_N^{k, \mathbb{B}}$, which itself is a consequence of the *injectivity* of the map $\Gamma_N^{k, \mathbb{B}}$, see (3), for N larger than some N_0 . We give here an alternative proof of this fact, which yields a quantitative upper bound on N_0 , and that will be useful later.

3.2. The injectivity and the intersection property. For any $n, k, m \in \mathbb{N}$ and $\mathbb{B} \in \text{Mat}_{1,n}(\text{Mat}_k(\mathbb{C}))$, let $\mathcal{K}_m(\mathbb{B})$ be the following span of monomials of degree m in the B_μ 's,

$$(6) \quad \mathcal{K}_m(\mathbb{B}) := \text{span} \{ B_{\mu_m} B_{\mu_{m-1}} \cdots B_{\mu_1} \mid (\mu_1, \dots, \mu_m) \subset \{1, \dots, n\}^{\times m} \}.$$

Furthermore, let

$$(7) \quad X_{n,k,m} := \{ \mathbb{B} \in \text{Mat}_{1,n}(\text{Mat}_k(\mathbb{C})) \mid \mathcal{K}_m(\mathbb{B}) = \text{Mat}_k(\mathbb{C}) \}.$$

Lemma 3.2. *Let $k \in \mathbb{N}$, and $\mathbb{B} \in \text{Mat}_{1,n}(\text{Mat}_k(\mathbb{C}))$. Then the followings are equivalent.*

- (i) $r_{\mathbb{B}} > 0$ and $r_{\mathbb{B}}^{-\frac{1}{2}} \mathbb{B} \in B_{n,k}$,
- (ii) there exists an $m \in \mathbb{N}$ such that $\mathbb{B} \in X_{n,k,m'}$, for all $m' \geq m$,
- (iii) there exists an $m \in \mathbb{N}$ such that $\mathbb{B} \in X_{n,k,m}$.

Furthermore, if these (equivalent) conditions hold, set

$$s^{k,\mathbb{B}} := \min \{ m \in \mathbb{N} \mid \mathbb{B} \in X_{n,k,m} \}.$$

Then,

- (a) for any $s \geq s^{k,\mathbb{B}}$, $\mathcal{K}_s(\mathbb{B}) = \text{Mat}_k(\mathbb{C})$,
- (b) for any $s \geq s^{k,\mathbb{B}}$, $\Gamma_s^{k,\mathbb{B}}$ is injective, and (k, \mathbb{B}) satisfies Property (I, $s+1$),
- (c) $s^{k,\mathbb{B}} \leq k^4$
- (d) if B_1 is invertible, then $s^{k,\mathbb{B}} \leq k^2$.

Proof. To prove the first part, we note that any of (i), (ii), (iii) implies the irreducibility of $\widehat{\mathbb{E}}^{\mathbb{B}}$. For if (i) holds, then for any orthogonal projection $P \in \mathcal{P}(\text{Mat}_k(\mathbb{C}))$ with $\widehat{\mathbb{E}}^{\mathbb{B}}(P \text{Mat}_k(\mathbb{C}) P) \subset P \text{Mat}_k(\mathbb{C}) P$, we have

$$\varphi_{(r_{\mathbb{B}})^{-1} \widehat{\mathbb{E}}^{\mathbb{B}}}(P) e_{(r_{\mathbb{B}})^{-1} \widehat{\mathbb{E}}^{\mathbb{B}}} = \lim_{N \rightarrow \infty} r_{\mathbb{B}}^{-N} (\widehat{\mathbb{E}}^{\mathbb{B}})^N(P) \in P \text{Mat}_k(\mathbb{C}) P.$$

As $(r_{\mathbb{B}})^{-1} \widehat{\mathbb{E}}^{\mathbb{B}} \in \mathcal{T}_k$, if $P \neq 0$, the left hand side is strictly positive, hence $P = 1$. This means the irreducibility of $\widehat{\mathbb{E}}^{\mathbb{B}}$. If (iii) holds, it is easy to check that $(\widehat{\mathbb{E}}^{\mathbb{B}})^m(A) > 0$ for any nonzero $A \in \text{Mat}_k(\mathbb{C})_+$. Hence, for any $t > 0$ and nonzero $A \in \text{Mat}_k(\mathbb{C})_+$, we have $\exp(t \widehat{\mathbb{E}}^{\mathbb{B}})(A) > 0$, hence $\widehat{\mathbb{E}}^{\mathbb{B}}$ is irreducible (see Theorem C.1). As (ii) implies (iii), this case is clear.

Therefore, throughout the proof, we may assume that $\widehat{\mathbb{E}}^{\mathbb{B}}$ is irreducible. In this case, the spectral radius $r_{\mathbb{B}}$ is strictly positive and a nondegenerate eigenvalue of $\widehat{\mathbb{E}}^{\mathbb{B}}$ with some strictly positive eigenvector $h_{\mathbb{B}}$ (see Theorem C.3). Hence,

$$\widehat{T}_{\mathbb{B}} := (r_{\mathbb{B}})^{-1} h_{\mathbb{B}}^{-\frac{1}{2}} \widehat{\mathbb{E}}^{\mathbb{B}} \left(h_{\mathbb{B}}^{\frac{1}{2}}(\cdot) h_{\mathbb{B}}^{\frac{1}{2}} \right) h_{\mathbb{B}}^{-\frac{1}{2}}$$

is a well-defined unital completely positive map. Furthermore, we set

$$\bar{\mathbb{B}} := \left((r_{\mathbb{B}})^{-\frac{1}{2}} h_{\mathbb{B}}^{-\frac{1}{2}} B_1 h_{\mathbb{B}}^{\frac{1}{2}}, (r_{\mathbb{B}})^{-\frac{1}{2}} h_{\mathbb{B}}^{-\frac{1}{2}} B_2 h_{\mathbb{B}}^{\frac{1}{2}}, \dots, (r_{\mathbb{B}})^{-\frac{1}{2}} h_{\mathbb{B}}^{-\frac{1}{2}} B_n h_{\mathbb{B}}^{\frac{1}{2}} \right),$$

and note that $\widehat{T}_{\mathbb{B}} = \widehat{\mathbb{E}}^{\bar{\mathbb{B}}}$.

By Theorem C.4, the followings are equivalent:

- (i)' There exists a unique faithful $\widehat{T}_{\mathbb{B}}$ -invariant state $\psi_{\mathbb{B}}$, and it satisfies

$$\lim_{l \rightarrow \infty} (\widehat{T}_{\mathbb{B}})^l(A) = \psi_{\mathbb{B}}(A) 1, \quad A \in \text{Mat}_k(\mathbb{C}),$$

- (ii)' there exists an $m \in \mathbb{N}$ such that $\mathcal{K}_{m'}(\bar{\mathbb{B}}) = \text{Mat}_k(\mathbb{C})$, for all $m' \geq m$,
- (iii)' there exists an $m \in \mathbb{N}$ such that $\mathcal{K}_m(\bar{\mathbb{B}}) = \text{Mat}_k(\mathbb{C})$.

Clearly, (ii), resp. (iii), is equivalent to (ii)', resp. (iii)'. Hence it suffices to show that (i) is equivalent to (i)'.

As (i) implies $(r_{\mathbb{B}})^{-1} \widehat{\mathbb{E}}^{\mathbb{B}} \in \mathcal{T}_k$, there exists a $(r_{\mathbb{B}})^{-1} \widehat{\mathbb{E}}^{\mathbb{B}}$ -invariant faithful state $\varphi_{(r_{\mathbb{B}})^{-\frac{1}{2}} \mathbb{B}}$ on $\text{Mat}_k(\mathbb{C})$, and we have

$$\lim_{N \rightarrow \infty} (\widehat{T}_{\mathbb{B}})^N(A) = \psi_{\mathbb{B}}(A) 1, \quad A \in \text{Mat}_k(\mathbb{C}).$$

Here,

$$\psi_{\mathbb{B}}(A) = \frac{\varphi_{(r_{\mathbb{B}})^{-\frac{1}{2}} \mathbb{B}} \left(h_{\mathbb{B}}^{\frac{1}{2}} A h_{\mathbb{B}}^{\frac{1}{2}} \right)}{\varphi_{(r_{\mathbb{B}})^{-\frac{1}{2}} \mathbb{B}}(h_{\mathbb{B}})}, \quad A \in \text{Mat}_k(\mathbb{C})$$

is the unique $\widehat{T}_{\mathbb{B}}$ -invariant faithful state.

On the other hand, if (i)' holds, then as $\widehat{T}_{\mathbb{B}}$ is similar to $(r_{\mathbb{B}})^{-1}\widehat{\mathbb{E}}^{\mathbb{B}}$, the spectral radius of $(r_{\mathbb{B}})^{-1}\widehat{\mathbb{E}}^{\mathbb{B}}$ is 1, and it is a strictly positive non degenerate eigenvalue of $(r_{\mathbb{B}})^{-1}\widehat{\mathbb{E}}^{\mathbb{B}}$. Furthermore, we have $\sigma((r_{\mathbb{B}})^{-1}\widehat{\mathbb{E}}^{\mathbb{B}}) \setminus \{1\} = \sigma(\widehat{T}_{\mathbb{B}}) \setminus \{1\} \subset \{z \in \mathbb{C} : |z| < 1\}$, where the last inclusion is by the primitivity of the unital completely positive map $\widehat{T}_{\mathbb{B}}$ (see Theorem C.4). Hence we have $(r_{\mathbb{B}})^{-1}\widehat{\mathbb{E}}^{\mathbb{B}} \in \mathcal{T}_k$. By the similarity of $\widehat{T}_{\mathbb{B}}$ with $(r_{\mathbb{B}})^{-1}\widehat{\mathbb{E}}^{\mathbb{B}}$, we have

$$P_{\{1\}}^{(r_{\mathbb{B}})^{-1}\widehat{\mathbb{E}}^{\mathbb{B}}}(A) = \psi_{\mathbb{B}}\left(h_{\mathbb{B}}^{-\frac{1}{2}}Ah_{\mathbb{B}}^{-\frac{1}{2}}\right)h_{\mathbb{B}}, \quad A \in \text{Mat}_k(\mathbb{C}).$$

From this, $e_{(r_{\mathbb{B}})^{-1}\widehat{\mathbb{E}}^{\mathbb{B}}} = P_{\{1\}}^{(r_{\mathbb{B}})^{-1}\widehat{\mathbb{E}}^{\mathbb{B}}}(1) = \psi_{\mathbb{B}}(h_{\mathbb{B}}^{-1})h_{\mathbb{B}}$, is invertible in $\text{Mat}_k(\mathbb{C})$. Furthermore, we have

$$P_{\{1\}}^{(r_{\mathbb{B}})^{-1}\widehat{\mathbb{E}}^{\mathbb{B}}}(A) = \frac{\psi_{\mathbb{B}}(h_{\mathbb{B}}^{-\frac{1}{2}}Ah_{\mathbb{B}}^{-\frac{1}{2}})}{\psi_{\mathbb{B}}(h_{\mathbb{B}}^{-1})}e_{(r_{\mathbb{B}})^{-1}\widehat{\mathbb{E}}^{\mathbb{B}}}, \quad A \in \text{Mat}_k(\mathbb{C}).$$

Here, the state

$$\varphi_{(r_{\mathbb{B}})^{-\frac{1}{2}}\mathbb{B}} = \frac{\psi_{\mathbb{B}}(h_{\mathbb{B}}^{-\frac{1}{2}} \cdot h_{\mathbb{B}}^{-\frac{1}{2}})}{\psi_{\mathbb{B}}(h_{\mathbb{B}}^{-1})}$$

is faithful on $\text{Mat}_k(\mathbb{C})$, because $\psi_{\mathbb{B}}$ is faithful.

Next we consider the latter half. To see (a), assume that $\mathcal{K}_s(\mathbb{B}) \neq \text{Mat}_k(\mathbb{C})$ for some $s > s^{k,\mathbb{B}}$. This means that there exists a nonzero $A \in \text{Mat}_k(\mathbb{C})$ such that $\text{Tr}_{\text{Mat}_k(\mathbb{C})}(A^*B_{\mu_s} \cdots B_{\mu_1}) = 0$ for any $\mu_1, \dots, \mu_s \in \{1, \dots, n\}$. As $\mathcal{K}_{s^{k,\mathbb{B}}}(\mathbb{B}) = \text{Mat}_k(\mathbb{C})$, this implies $A^*B_{\mu_s} \cdots B_{\mu_{s^{k,\mathbb{B}}+1}} = 0$, for any $\mu_{s^{k,\mathbb{B}}+1}, \dots, \mu_s \in \{1, \dots, n\}$. Hence we have $0 = A^*(\widehat{\mathbb{E}})^{s-s^{k,\mathbb{B}}}(e_{\mathbb{B}}) = (r_{\mathbb{B}})^{s-s^{k,\mathbb{B}}}A^*e_{\mathbb{B}}$. Because $e_{\mathbb{B}}$ is invertible, this means $A = 0$, a contradiction.

To see the injectivity stated in (b) for $s \geq s^{k,\mathbb{B}}$, suppose that $\Gamma_s^{k,\mathbb{B}}(C) = 0$, for $C \in \text{Mat}_k(\mathbb{C})$. Then, $\text{Tr}_{\text{Mat}_k(\mathbb{C})}(CB_{\mu(s)}^*) = 0$, for any s -tuple $\mu(s) \in \{1, \dots, n\}^{\times s}$. As $s \geq s^{k,\mathbb{B}}$, $\mathcal{K}_s(\mathbb{B}) = \text{Mat}_k(\mathbb{C})$ by (a), hence this implies $C = 0$, and the injectivity holds. Property (I, $s+1$) in (b) can then be checked as in [FNW]. Finally, (c), (d) is the quantum Wielandt's inequality in the least optimal case of only one linearly independent Kraus operator [SPWC]. \square

3.3. Edge ground states. Now we consider in detail the edge ground states, namely the half-chain analog of [FNW]. Throughout this section we fix an orthonormal basis $\{e_{\alpha}\}_{\alpha=1}^k$ of \mathbb{C}^k . Furthermore, we define a sesquilinear form $\langle, \rangle_{\mathbb{B}}$ on $\text{Mat}_k(\mathbb{C})$ by

$$\langle B, C \rangle_{\mathbb{B}} := \varphi^{\mathbb{B}}(B^*e_{\mathbb{B}}C), \quad B, C \in \text{Mat}_k(\mathbb{C}).$$

As $\varphi^{\mathbb{B}}$ is faithful on $\text{Mat}_k(\mathbb{C})$ and $e_{\mathbb{B}}$ is invertible in $\text{Mat}_k(\mathbb{C})$, this gives an inner product on $\text{Mat}_k(\mathbb{C})$. Furthermore,

$$(8) \quad \text{Tr}_{\text{Mat}_k(\mathbb{C})}(X^*X) \leq a^{\mathbb{B}}c^{\mathbb{B}}\langle X, X \rangle_{\mathbb{B}}, \quad \text{Tr}_{\text{Mat}_k(\mathbb{C})}(X^*e_{\mathbb{B}}X) \leq c^{\mathbb{B}}\langle X, X \rangle_{\mathbb{B}}, \quad \varphi^{\mathbb{B}}(X^*X) \leq a^{\mathbb{B}}\langle X, X \rangle_{\mathbb{B}},$$

for any $X \in \text{Mat}_k(\mathbb{C})$.

The following estimate can be found in [FNW]. We repeat its proof for completeness here.

Lemma 3.3. *Let $\mathbb{B} \in B_{n,k}$. Then for any $B, C \in \text{Mat}_k(\mathbb{C})$ and $N \in \mathbb{N}$,*

$$\left| \left\langle \Gamma_N^{k,\mathbb{B}}(B), \Gamma_N^{k,\mathbb{B}}(C) \right\rangle - \langle B, C \rangle_{\mathbb{B}} \right| \leq E^{k,\mathbb{B}}(N) \langle B, B \rangle_{\mathbb{B}}^{1/2} \langle C, C \rangle_{\mathbb{B}}^{1/2}.$$

Here \langle, \rangle indicates the inner product of $\bigotimes_{i=0}^{N-1} \mathbb{C}^n$. In particular, for any $N \in \mathbb{N}$,

$$(9) \quad (1 - E^{k,\mathbb{B}}(N)) \langle B, B \rangle_{\mathbb{B}} \leq \left\| \Gamma_N^{k,\mathbb{B}}(B) \right\|^2 \leq (1 + E^{k,\mathbb{B}}(N)) \langle B, B \rangle_{\mathbb{B}}, \quad B \in \text{Mat}_k(\mathbb{C}),$$

and $\Gamma_N^{k,\mathbb{B}}$ is injective if $E^{k,\mathbb{B}}(N) < 1$.

Proof. First, recall that $P_{\{1\}}^{\mathbb{B}}(a) = \varphi^{\mathbb{B}}(a)e_{\mathbb{B}}$. Then

$$\left\langle \Gamma_{N,p,q}^{k,\mathbb{B}}(B), \Gamma_{N,p,q}^{k,\mathbb{B}}(C) \right\rangle = \sum_{\alpha, \beta=1}^k \left[\left\langle e_{\alpha}, (\widehat{\mathbb{E}}^{\mathbb{B}})^N (1 - P_{\{1\}}^{\mathbb{B}})(B^*|e_{\alpha}\rangle\langle e_{\beta}|C)e_{\beta} \right\rangle + \varphi^{\mathbb{B}}(B^*|e_{\alpha}\rangle\langle e_{\beta}|C)\langle e_{\alpha}, e_{\mathbb{B}}e_{\beta} \rangle \right].$$

The second term is equal to $\varphi^{\mathbb{B}}(B^*e_{\mathbb{B}}C) = \langle B, C \rangle_{\mathbb{B}}$. The first term can be bounded from above by

$$\sum_{\alpha, \beta=1}^k \|e_{\alpha}\| \|(\widehat{\mathbb{E}}^{\mathbb{B}})^{(N)}(1-P_{\{1\}}^{\mathbb{B}})\| \|B^*e_{\alpha}\| \|C^*e_{\beta}\| \|e_{\beta}\| \leq k \|(\widehat{\mathbb{E}}^{\mathbb{B}})^{(N)}(1-P_{\{1\}}^{\mathbb{B}})\| \operatorname{Tr}_{\operatorname{Mat}_k(\mathbb{C})}(B^*B)^{1/2} \operatorname{Tr}_{\operatorname{Mat}_k(\mathbb{C})}(C^*C)^{1/2}$$

using the Cauchy-Schwarz inequality for \mathbb{C}^k . The first part of the lemma follows from this combined with the observation (8). The second inequality (9) can be immediately checked from the first inequality. Finally, if $E_{p,q}^{k,\mathbb{B}}(N) < 1$, then from (9), $\Gamma_{N,p,q}^{k,\mathbb{B}}(B) = 0$ implies $\langle B, B \rangle_{\mathbb{B}} = 0$. As $\langle \cdot, \cdot \rangle_{\mathbb{B}}$ is an inner product, this means $B = 0$. Therefore, $\Gamma_{N,p,q}^{k,\mathbb{B}}$ is injective. \square

For integers $b \leq a$ and an $a-b+1$ -tuple $\mu^{(a-b+1)} = (\mu_1^{(a-b+1)}, \dots, \mu_{a-b+1}^{(a-b+1)}) \in \{1, \dots, n\}^{\times(a-b+1)}$, $\bar{\psi}_{\mu^{(a-b+1)}}^{[b,a]}$ indicates the vector $\psi_{\mu_1^{(a-b+1)}} \otimes \psi_{\mu_2^{(a-b+1)}} \otimes \dots \otimes \psi_{\mu_{a-b+1}^{(a-b+1)}}$ in $\otimes_{i=b}^a \mathbb{C}^n$.

Lemma 3.4. *Let $\mathbb{B} \in B_{n,k}$ and let $R^{\mathbb{B}} : \mathcal{A}_{[0,\infty)}^{\operatorname{loc}} \rightarrow \operatorname{Mat}_k(\mathbb{C})$ be defined by*

$$R^{\mathbb{B}}(A) := \sum_{\mu^{(a)}, \nu^{(a)} \in \{1, \dots, n\}^{\times a}} \left\langle \bar{\psi}_{\mu^{(a)}}^{[0,a-1]}, A \bar{\psi}_{\nu^{(a)}}^{[0,a-1]} \right\rangle B_{\mu^{(a)}} e_{\mathbb{B}} B_{\nu^{(a)}}^*,$$

if $A \in \mathcal{A}_{[0,a-1]}$ for $a \in \mathbb{N}$. Then $R^{\mathbb{B}}$ is well-defined and extends to a completely positive map from the half-infinite chain $\mathcal{A}_{[0,\infty)}$ onto $\operatorname{Mat}_k(\mathbb{C})$, which we will denote by the same symbol $R^{\mathbb{B}}$. Furthermore, for any $A \in \mathcal{A}_{[0,\infty)}^{\operatorname{loc}}$ and $C \in \operatorname{Mat}_k(\mathbb{C})$, we have

$$\lim_{N \rightarrow \infty} \left\langle \Gamma_N^{k,\mathbb{B}}(C), A \Gamma_N^{k,\mathbb{B}}(C) \right\rangle = \varphi^{\mathbb{B}}(C^* R^{\mathbb{B}}(A) C).$$

Proof. Note that from the relation $\sum_{\mu=1}^n B_{\mu} e_{\mathbb{B}} B_{\mu}^* = e_{\mathbb{B}}$, $R^{\mathbb{B}}$ is well-defined. It can be checked directly that $R^{\mathbb{B}}|_{\mathcal{A}_{[0,a-1]}}$ defines a completely positive map on $\mathcal{A}_{[0,a-1]}$ with norm $\|R^{\mathbb{B}}|_{\mathcal{A}_{[0,a-1]}}\| = \|R^{\mathbb{B}}(1)\| = \|e_{\mathbb{B}}\|$, for any $a \in \mathbb{N}$. Therefore, we can extend it to a completely positive map on $\mathcal{A}_{[0,\infty)}$.

To see that $R^{\mathbb{B}}$ is surjective, recall from Lemma 3.2, that for $\mathbb{B} \in B_{n,k}$ and $s^{k,\mathbb{B}} \leq a \in \mathbb{N}$, we have $\mathcal{K}_a(\mathbb{B}) = \operatorname{Mat}_k(\mathbb{C})$. Therefore, for any $\xi_1, \xi_2 \in \mathbb{C}^k$ and $\eta \in p\mathbb{C}^k \setminus \{0\}$, there exist $\{\alpha_{\mu^{(a)}}\}_{\mu^{(a)} \in \{1, \dots, n\}^{\times a}} \subset \mathbb{C}$ and $\{\beta_{\nu^{(a)}}\}_{\nu^{(a)} \in \{1, \dots, n\}^{\times a}} \subset \mathbb{C}$

$$\sum_{\mu^{(a)}} \alpha_{\mu^{(a)}} B_{\mu^{(a)}} = \frac{1}{\langle \eta, e_{\mathbb{B}} \eta \rangle} |\xi_1\rangle \langle \eta|, \quad \sum_{\nu^{(a)}} \beta_{\nu^{(a)}} B_{\nu^{(a)}}^* = |\eta\rangle \langle \xi_2|.$$

Then for

$$A = \sum_{\mu^{(a)}, \nu^{(a)}} \alpha_{\mu^{(a)}} \beta_{\nu^{(a)}} \left| \bar{\psi}_{\mu^{(a)}}^{[0,a-1]} \right\rangle \left\langle \bar{\psi}_{\nu^{(a)}}^{[0,a-1]} \right|,$$

we have $R^{\mathbb{B}}(A) = |\xi_1\rangle \langle \xi_2|$. Hence, $R^{\mathbb{B}}$ is surjective.

For the latter equality, if $A \in \mathcal{A}_{[0,a-1]}$, as in Lemma 3.3, we have

$$\begin{aligned} & \left\langle \Gamma_N^{k,\mathbb{B}}(C), A \Gamma_N^{k,\mathbb{B}}(C) \right\rangle \\ &= \sum_{\substack{\mu^{(a)}, \nu^{(a)} \in \{1, \dots, n\}^{\times a} \\ \mu^{(N-a)} \in \{1, \dots, n\}^{\times(N-a)}}} \left\langle \bar{\psi}_{\mu^{(a)}}^{[0,a-1]}, A \bar{\psi}_{\nu^{(a)}}^{[0,a-1]} \right\rangle \operatorname{Tr}_{\operatorname{Mat}_k(\mathbb{C})}(C^* B_{\mu^{(a)}} B_{\mu^{(N-a)}}) \operatorname{Tr}_{\operatorname{Mat}_k(\mathbb{C})}(B_{\mu^{(N-a)}}^* B_{\nu^{(a)}}^* C) \\ &= \sum_{\alpha, \beta=1}^k \sum_{\mu^{(a)}, \nu^{(a)} \in \{1, \dots, n\}^{\times a}} \left\langle e_{\alpha}, \left(\widehat{\mathbb{E}}^{\mathbb{B}}\right)^{N-a} (C^* B_{\mu^{(a)}} |e_{\alpha}\rangle \langle e_{\beta}| B_{\nu^{(a)}}^* C) e_{\beta} \right\rangle \left\langle \bar{\psi}_{\mu^{(a)}}^{[0,a-1]}, A \bar{\psi}_{\nu^{(a)}}^{[0,a-1]} \right\rangle \\ &\rightarrow \sum_{\alpha, \beta=1}^k \sum_{\mu^{(a)}, \nu^{(a)} \in \{1, \dots, n\}^{\times a}} \langle e_{\alpha}, e_{\mathbb{B}} e_{\beta} \rangle \varphi^{\mathbb{B}}(C^* B_{\mu^{(a)}} |e_{\alpha}\rangle \langle e_{\beta}| B_{\nu^{(a)}}^* C) \left\langle \bar{\psi}_{\mu^{(a)}}^{[0,a-1]}, A \bar{\psi}_{\nu^{(a)}}^{[0,a-1]} \right\rangle \\ &= \sum_{\mu^{(a)}, \nu^{(a)} \in \{1, \dots, n\}^{\times a}} \varphi^{\mathbb{B}}(C^* B_{\mu^{(a)}} e_{\mathbb{B}} B_{\nu^{(a)}}^* C) \left\langle \bar{\psi}_{\mu^{(a)}}^{[0,a-1]}, A \bar{\psi}_{\nu^{(a)}}^{[0,a-1]} \right\rangle = \varphi^{\mathbb{B}}(C^* R^{\mathbb{B}}(A) C). \end{aligned}$$

□

Similarly,

Lemma 3.5. *Let $\mathbb{B} \in B_{n,k}$ and let $L^{\mathbb{B}} : \mathcal{A}_{(-\infty, -1]}^{\text{loc}} \rightarrow \text{Mat}_k(\mathbb{C})$ be defined by*

$$L^{\mathbb{B}}(A) := \sum_{\mu^{(b)}, \nu^{(b)} \in \{1, \dots, n\} \times b} \left\langle \bar{\psi}_{\mu^{(b)}}^{[-b, -1]}, A \bar{\psi}_{\nu^{(b)}}^{[-b, -1]} \right\rangle B_{\nu^{(b)}}^* \rho^{\mathbb{B}} B_{\mu^{(b)}},$$

if $A \in \mathcal{A}_{[-b, -1]}$ for $b \in \mathbb{N}$. Then $L^{\mathbb{B}}$ is well-defined and extends to a completely positive map from the half-infinite chain $\mathcal{A}_{(-\infty, -1]}$ onto $\text{Mat}_k(\mathbb{C})$, which we will denote by the same symbol $L^{\mathbb{B}}$. Furthermore, for any $A \in \mathcal{A}_{(-\infty, -1]}^{\text{loc}}$ and $C \in \text{Mat}_k(\mathbb{C})$, we have

$$\lim_{N \rightarrow \infty} \left\langle \Gamma_N^{k, \mathbb{B}}(C), \tau_N(A) \Gamma_N^{k, \mathbb{B}}(C) \right\rangle = \text{Tr}_{\text{Mat}_k(\mathbb{C})} (e_{\mathbb{B}} C L^{\mathbb{B}}(A) C^*).$$

If ω is a state on $\text{Mat}_k(\mathbb{C})$,

$$\omega_R^{\mathbb{B}}(A) := \omega \left(e_{\mathbb{B}}^{-1/2} R^{\mathbb{B}}(A) e_{\mathbb{B}}^{-1/2} \right), \quad A \in \mathcal{A}_{[0, \infty)},$$

defines a state on $\mathcal{A}_{[0, \infty)}$. Similarly, if ω is a state on $\text{Mat}_k(\mathbb{C})$, then

$$\omega_L^{\mathbb{B}}(A) := \omega \left((\rho^{\mathbb{B}})^{-1/2} L^{\mathbb{B}}(A) (\rho^{\mathbb{B}})^{-1/2} \right), \quad A \in \mathcal{A}_{(-\infty, -1]},$$

defines a state on $\mathcal{A}_{(-\infty, -1]}$.

We denote the sets of these states by

$$(10) \quad \mathcal{E}_R^{k, \mathbb{B}} := \{\omega_R^{\mathbb{B}} : \omega \text{ is a state on } \text{Mat}_k(\mathbb{C})\},$$

$$(11) \quad \mathcal{E}_L^{k, \mathbb{B}} := \{\omega_L^{\mathbb{B}} : \omega \text{ is a state on } \text{Mat}_k(\mathbb{C})\}.$$

Recall that \mathfrak{E}_d denotes the set of states on $\text{Mat}_d(\mathbb{C})$ for $d \in \mathbb{N}$.

Lemma 3.6. *Let $k \in \mathbb{N}$ and $\mathbb{B} \in B_{n,k}$. Then the maps $\Xi_R : \mathfrak{E}_k \rightarrow \mathcal{E}_R^{k, \mathbb{B}}$, $\Xi_L : \mathfrak{E}_k \rightarrow \mathcal{E}_L^{k, \mathbb{B}}$ defined by*

$$\Xi_R(\omega) := \omega_R^{\mathbb{B}}, \quad \Xi_L(\omega) := \omega_L^{\mathbb{B}},$$

are affine bijections.

Proof. From the definition, it is clear that Ξ_L, Ξ_R are affine surjection. As $R^{\mathbb{B}}$ (resp. $L^{\mathbb{B}}$) is surjective and $e_{\mathbb{B}}$ (resp. $\rho^{\mathbb{B}}$) is bijective in $\text{Mat}_k(\mathbb{C})$, Ξ_R (resp. Ξ_L) is injective. □

As in [FNW], we can show the following:

Lemma 3.7. *Let $\mathbb{B} \in B_{n,k}$ and let $\omega_{\infty}^{\mathbb{B}} : \mathcal{A}_{\mathbb{Z}}^{\text{loc}} \rightarrow \mathbb{C}$ be defined by*

$$\omega_{\infty}^{\mathbb{B}}(A) := \sum_{\mu^{(a-b+1)}, \nu^{(a-b+1)} \in \{1, \dots, n\} \times a-b+1} \left\langle \bar{\psi}_{\mu^{(a-b+1)}}^{[b, a]}, A \bar{\psi}_{\nu^{(a-b+1)}}^{[b, a]} \right\rangle \varphi^{\mathbb{B}}(B_{\mu^{(a-b+1)}} e_{\mathbb{B}} B_{\nu^{(a-b+1)}}^*)$$

if $A \in \mathcal{A}_{[b, a]}$ with $b \leq a$. Then $\omega_{\infty}^{\mathbb{B}}$ is well-defined and extends to a state on the chain $\mathcal{A}_{\mathbb{Z}}$. Furthermore, for any $A \in \mathcal{A}_{\mathbb{Z}}^{\text{loc}}$, and $C \in \text{Mat}_k(\mathbb{C})$, we have

$$\lim_{N, M \rightarrow \infty} \left\langle \Gamma_{N+M}^{k, \mathbb{B}}(C), \tau_N(A) \Gamma_{N+M}^{k, \mathbb{B}}(C) \right\rangle = \omega_{\infty}^{\mathbb{B}}(A) \varphi^{\mathbb{B}}(C^* e_{\mathbb{B}} C).$$

On the other hand, for $\Phi_m^{k, \mathbb{B}} = 1 - G_m^{k, \mathbb{B}}$, $m \in \mathbb{N}$, recall that $\mathcal{S}_{[0, \infty)}(H_m^{k, \mathbb{B}})$, $\mathcal{S}_{(-\infty, -1]}(H_m^{k, \mathbb{B}})$, $\mathcal{S}_{\mathbb{Z}}(H_m^{k, \mathbb{B}})$ are the set of all wk^* -accumulation points of ground states in finite intervals.

Lemma 3.8. *Let $k \in \mathbb{N}$ and $\mathbb{B} \in B_{n,k}$. Then for $m \geq m^{k, \mathbb{B}}$,*

$$\mathcal{S}_{[0, \infty)}(H_m^{k, \mathbb{B}}) = \{\omega : \omega \text{ is a state on } \mathcal{A}_{[0, \infty)} \text{ such that } \omega \circ \tau_a(\Phi_m^{k, \mathbb{B}}) = 0, \text{ for all } 0 \leq a \in \mathbb{Z}\}$$

$$\mathcal{S}_{(-\infty, -1]}(H_m^{k, \mathbb{B}}) = \{\omega : \omega \text{ is a state on } \mathcal{A}_{(-\infty, -1]} \text{ such that } \omega \circ \tau_{-b}(\Phi_m^{k, \mathbb{B}}) = 0, \text{ for all } m \leq b \in \mathbb{Z}\},$$

$$\mathcal{S}_{\mathbb{Z}}(H_m^{k, \mathbb{B}}) = \{\omega : \omega \text{ is a state on } \mathcal{A}_{\mathbb{Z}} \text{ such that } \omega \circ \tau_a(\Phi_m^{k, \mathbb{B}}) = 0, \text{ for all } a \in \mathbb{Z}\},$$

Proof. If ω is a state on $\mathcal{A}_{[0,\infty)}$ such that $\omega \circ \tau_a(\Phi_m^{k,\mathbb{B}}) = 0$, for all $0 \leq a \in \mathbb{Z}$. Then its restriction $\omega|_{\mathcal{A}_\Lambda}$ to each interval $\Lambda \subset [0, \infty)$ is a ground state of $(H_m^{k,\mathbb{B}})_\Lambda$. Hence ω is a wk^* -accumulation point of extensions of $\omega|_{\mathcal{A}_\Lambda} \in \mathcal{S}_\Lambda((H_m^{k,\mathbb{B}})_\Lambda)$, hence $\omega \in \mathcal{S}_{[0,\infty)}(H_m^{k,\mathbb{B}})$, by definition. On the other hand, if $\omega \in \mathcal{S}_{[0,\infty)}(H_m^{k,\mathbb{B}})$, then there exists a subnet $\{\Lambda'\}$ of intervals in $[0, \infty)$ associated with states $\omega_{\Lambda'}$ on $\mathcal{A}_{[0,\infty)}$ such that $\omega_{\Lambda'}|_{\mathcal{A}_{\Lambda'}} \in \mathcal{S}_{\Lambda'}((H_m^{k,\mathbb{B}})_{\Lambda'})$, and $\omega = \text{wk}^* - \lim_{\Lambda'} \omega_{\Lambda'}$. Hence we have $\omega \circ \tau_a(\Phi_m^{k,\mathbb{B}}) = \lim_{\Lambda'} \omega_{\Lambda'}(\tau_a(\Phi_m^{k,\mathbb{B}})) = 0$, for all $0 \leq a \in \mathbb{Z}$. \square

Proposition 3.9. *Let $k \in \mathbb{N}$ and $\mathbb{B} \in B_{n,k}$. Then for $m \geq m^{k,\mathbb{B}}$,*

$$\mathcal{S}_{[0,\infty)}(H_m^{k,\mathbb{B}}) = \mathcal{E}_R^{k,\mathbb{B}}, \quad \mathcal{S}_{(-\infty,-1]}(H_m^{k,\mathbb{B}}) = \mathcal{E}_L^{k,\mathbb{B}}, \quad \mathcal{S}_{\mathbb{Z}}(H_m^{k,\mathbb{B}}) = \{\omega_\infty^{\mathbb{B}}\},$$

where $\mathcal{E}_R^{k,\mathbb{B}}, \mathcal{E}_L^{k,\mathbb{B}}$ were defined in (10,11). In particular, the maps

$$\Xi_R : \mathfrak{E}_k \rightarrow \mathcal{E}_R^{k,\mathbb{B}} = \mathcal{S}_{[0,\infty)}(H_m^{k,\mathbb{B}}), \quad \Xi_L : \mathfrak{E}_k \rightarrow \mathcal{E}_L^{k,\mathbb{B}} = \mathcal{S}_{(-\infty,-1]}(H_m^{k,\mathbb{B}})$$

are affine bijections.

Proof. First we show $\mathcal{E}_R^{k,\mathbb{B}} \subset \mathcal{S}_{[0,\infty)}(H_m^{k,\mathbb{B}})$. Let ω be a state on $\text{Mat}_k(\mathbb{C})$ given by a density matrix

$$\sum_i |\xi_i\rangle \langle \xi_i|, \quad \xi_i \in p\mathbb{C}^k \setminus \{0\}.$$

Fix a nonzero $\eta \in \mathbb{C}^k$ and set $C_i = |\xi_i\rangle \langle \eta| \in \text{Mat}_k(\mathbb{C})$. A direct calculation shows that

$$\sum_i \|\xi_i\|^2 \sigma_{C_i} = \omega_R^{\mathbb{B}},$$

where $\sigma_B(A) = (\varphi^{\mathbb{B}}(B^*B))^{-1} \varphi^{\mathbb{B}}(B^*e_{\mathbb{B}}^{-1/2}R^{\mathbb{B}}(A)e_{\mathbb{B}}^{-1/2}B)$ defines a state on $A_{[0,\infty)}$ which belongs to $\mathcal{E}_R^{k,\mathbb{B}}$, for any nonzero $B \in \text{Mat}_k(\mathbb{C})$. Therefore, it suffices to show that $\sigma_B(\tau_a(\Phi_m^{k,\mathbb{B}})) = 0$ for all $0 \leq a$ and nonzero $B \in \text{Mat}_k(\mathbb{C})$. But this follows from

$$\sigma_B(\tau_a(\Phi_m^{k,\mathbb{B}})) = \lim_{N \rightarrow \infty} \frac{\left\langle \Gamma_N^{k,\mathbb{B}}(e_{\mathbb{B}}^{-1/2}B), \tau_a(\Phi_m^{k,\mathbb{B}}) \Gamma_N^{k,\mathbb{B}}(e_{\mathbb{B}}^{-1/2}B) \right\rangle}{\varphi^{\mathbb{B}}(B^*B)}$$

and the fact that by definition of $\Phi_m^{k,\mathbb{B}}$ and the intersection property the numerator on the right hand side is uniformly equal to 0 for $N \geq m + a - 1$.

Next we show $\mathcal{S}_{[0,\infty)}(H_m^{k,\mathbb{B}}) \subset \mathcal{E}_R^{k,\mathbb{B}}$. Let $\omega \in \mathcal{S}_{[0,\infty)}(H_m^{k,\mathbb{B}})$. For each $N \geq m^{k,\mathbb{B}}$, let D_N be the density matrix of the restriction of ω to $\mathcal{A}_{[0,N-1]}$, namely $\omega(A) = \text{Tr}_{[0,N-1]}(D_N A)$ for any $A \in \mathcal{A}_{[0,N-1]}$. By the condition $\omega(\tau_a(\Phi_m^{k,\mathbb{B}})) = 0$, $0 \leq a \leq N - m$, and the intersection property, we have that $\text{Ran}(D_N) \subset \mathcal{G}_N^{k,\mathbb{B}}$. Therefore, there exist $X_{i,N} \in \text{Mat}_k(\mathbb{C})$, $i = 1, \dots, k^2$ such that

$$(12) \quad D_N = \sum_i \left| \Gamma_N^{k,\mathbb{B}}(X_{i,N}) \right\rangle \left\langle \Gamma_N^{k,\mathbb{B}}(X_{i,N}) \right|.$$

Note that, using first Lemma 3.3 and then (8),

$$1 = \sum_i \|\Gamma_N^{k,\mathbb{B}}(X_{i,N})\|^2 \geq \sum_i \langle X_{i,N}, X_{i,N} \rangle_{\mathbb{B}} (1 - E^{k,\mathbb{B}}(N)) \geq \frac{1}{2} (a^{\mathbb{B}} c^{\mathbb{B}})^{-1} \sum_i \text{Tr}_{\text{Mat}_k(\mathbb{C})} (X_{i,N}^* X_{i,N}),$$

for $N \geq L^{k,\mathbb{B}}$. Hence, by compactness, there is a subsequence $\{N_m\}_m$ such that

$$\lim_{m \rightarrow \infty} X_{i,N_m} = X_{i,\infty}$$

for all i . From Lemma 3.3, we have

$$(13) \quad \limsup_{m \rightarrow \infty} \sum_i \|\Gamma_{N_m}^{k,\mathbb{B}}(X_{i,N_m} - X_{i,\infty})\|^2 \leq \limsup_{m \rightarrow \infty} \sum_i (1 + E^{k,\mathbb{B}}(N_m)) \langle X_{i,N_m} - X_{i,\infty}, X_{i,N_m} - X_{i,\infty} \rangle_{\mathbb{B}} = 0.$$

From this we have

$$(14) \quad 1 = \lim_{m \rightarrow \infty} \sum_i \|\Gamma_{N_m}^{k,\mathbb{B}}(X_{i,N_m})\|^2 = \lim_{m \rightarrow \infty} \sum_i \|\Gamma_{N_m}^{k,\mathbb{B}}(X_{i,\infty})\|^2 = \sum_i \langle X_{i,\infty}, X_{i,\infty} \rangle_{\mathbb{B}}.$$

Therefore, there exists a nonzero $X_{i,\infty}$.

Now, set $Y_i := e_{\mathbb{B}}^{1/2} X_{i,\infty}$. Note that $Y_i \neq 0$, if $X_{i,\infty} \neq 0$. The form $\tilde{\omega}$ on $\mathcal{A}_{[0,\infty)}$ defined by

$$\tilde{\omega} := \sum_{i: X_{i,\infty} \neq 0} \langle X_{i,\infty}, X_{i,\infty} \rangle_{\mathbb{B}} \sigma_{Y_i}$$

is a state by (14). Furthermore, $\tilde{\omega} \in \mathcal{E}_R^{k,\mathbb{B}}$, as each σ_{Y_i} is, and we have

$$\tilde{\omega}(A) = \sum_{i: X_{i,\infty} \neq 0} \varphi^{\mathbb{B}} \left(Y_i^* e_{\mathbb{B}}^{-1/2} R^{\mathbb{B}}(A) e_{\mathbb{B}}^{-1/2} Y_i \right) = \sum_{i: X_{i,\infty} \neq 0} \varphi^{\mathbb{B}} \left(X_{i,\infty}^* R^{\mathbb{B}}(A) X_{i,\infty} \right), \quad A \in \mathcal{A}_{[0,\infty)}.$$

We claim $\omega = \tilde{\omega}$: For $A \in \mathcal{A}^{\text{loc}}$, from (12), (13) and Lemma 3.4,

$$\begin{aligned} |\omega(A) - \tilde{\omega}(A)| &= \lim_{m \rightarrow \infty} \left| \sum_i \left(\langle \Gamma_{N_m}^{k,\mathbb{B}}(X_{i,N_m}), A \Gamma_{N_m}^{k,\mathbb{B}}(X_{i,N_m}) \rangle - \varphi^{\mathbb{B}}(X_{i,\infty}^* R^{\mathbb{B}}(A) X_{i,\infty}) \right) \right| \\ &= \lim_{m \rightarrow \infty} \left| \sum_i \left(\langle \Gamma_{N_m}^{k,\mathbb{B}}(X_{i,\infty}), A \Gamma_{N_m}^{k,\mathbb{B}}(X_{i,\infty}) \rangle - \varphi^{\mathbb{B}}(X_{i,\infty}^* R^{\mathbb{B}}(A) X_{i,\infty}) \right) \right| = 0, \end{aligned}$$

and in fact $\tilde{\omega} = \omega$. Hence, $\omega \in \mathcal{E}_R^{k,\mathbb{B}}$. The case $\mathcal{S}_{(-\infty,-1]}(H_m^{k,\mathbb{B}}) = \mathcal{E}_L^{k,\mathbb{B}}$ is treated similarly. \square

Proof of Proposition 2.4. The intersection property, part (i), follows from Lemma 3.2(b). Part(ii) is a consequence of Proposition 3.1. (iib) and (iic) were the contents of Proposition 3.9. \square

4. A CONTINUOUS PATH OF HAMILTONIANS

We shall write $H \simeq_C H'$ if the translation invariant Hamiltonians H and H' are C^1 -equivalent. For $m \in \mathbb{N}$ and translation invariant Hamiltonians H_m, H'_m with interaction length less than or equal to m , we further write $H_m \simeq_{C,m} H'_m$ if H_m and H'_m are C^1 -equivalent and the C^1 -path can be taken as a path in \mathcal{J}_m .

In this section, we prove the main result of this paper, Theorem 2.5. For now, we shall use the following technical result, which will be proved in Section 5, Proposition 5.1: If $\mathbb{B}, \mathbb{B}' \in B_{n,k}$ and for any $m \geq 2k(k-1)+3$, there exists a continuous map $\bar{\mathbb{A}} : [0,1] \rightarrow \text{Mat}_{1,n}(\text{Mat}_k(\mathbb{C}))$, piecewise of class C^1 , such that $\bar{\mathbb{A}}(0) = \mathbb{B}$, $\bar{\mathbb{A}}(1) = \mathbb{B}'$ and $\bar{\mathbb{A}}(t) \in X_{n,k,m}$ (see (7) for the definition of this set) with $\bar{\mathbb{A}}_1(t)$ invertible for $t \in (0,1)$.

With this, the proof of C^1 -equivalence relies on two results. Firstly, that given $\mathbb{B} \in B_{n,k}$ and two interactions of ranges m, m' such that $\mathcal{G}_m^{k,\mathbb{B}}$ and $\mathcal{G}_{m'}^{k,\mathbb{B}}$ satisfy the intersection property, then the two corresponding Hamiltonians are equivalent: We refer to this as equivalence under changing of the interaction length, Lemma 4.2. Secondly, if $\mathbb{B}, \mathbb{B}' \in B_{n,k}$ and $H_m^{k,\mathbb{B}}, H_m^{k,\mathbb{B}'}$ are the corresponding Hamiltonians (with the same m and k), then the smooth path of matrices $\bar{\mathbb{A}}(t)$ mentioned above yields the C^1 -equivalence of the Hamiltonians: This is the equivalence under smooth deformations of \mathbb{B} , Lemma 4.1.

We start with the equivalence under deformations of \mathbb{B} .

Lemma 4.1. *Let $k \in \mathbb{N}$ and $\mathbb{B}, \mathbb{B}' \in B_{n,k}$. Then for any $m \geq k^4 + 1$,*

$$H_m^{k,\mathbb{B}} \simeq_{C,m} H_m^{k,\mathbb{B}'}.$$

Proof. Let $\bar{\mathbb{A}}(t)$ be the path given by Proposition 5.1. By Lemma 3.2, $r_{\bar{\mathbb{A}}(t)} > 0$ for $t \in (0,1)$ and $(r_{\bar{\mathbb{A}}(t)})^{-\frac{1}{2}} \bar{\mathbb{A}}(t) \in B_{n,k}$. That $r_{\bar{\mathbb{A}}(t)} > 0$ and $\widehat{\mathbb{E}}^{\bar{\mathbb{A}}(t)} \in \mathcal{T}_k$ for $t = 0,1$ follows by definition.

Set $\mathbb{A}(t) := r_{\bar{\mathbb{A}}(t)}^{-\frac{1}{2}} \bar{\mathbb{A}}(t) \in B_{n,k}$, for $t \in [0,1]$. Applying Lemma D.2 to a continuous piecewise C^1 -path $[0,1] \ni t \mapsto \widehat{\mathbb{E}}^{\bar{\mathbb{A}}(t)} \in \mathcal{T}_k$, we see that $[0,1] \ni t \mapsto r_{\bar{\mathbb{A}}(t)}$ is continuous and piecewise C^1 . Therefore, the path $[0,1] \ni t \mapsto \widehat{\mathbb{E}}^{\mathbb{A}(t)} \in \mathcal{T}_k$ is continuous and piecewise C^1 .

Let $m_0 := k^4 + 1$. For any $m \geq m_0$, Lemma 3.2 ensure that $\Gamma_m^{k,\mathbb{A}(t)}$ is injective and that (k, \mathbb{B}) satisfies Property (I, m). Recall the definitions at the beginning of Section 3.1. We claim

- (i) $l_0 := \sup_{t \in [0,1]} \bar{l}^{k,\mathbb{A}(t)} < \infty$,

(ii) for any $m_0 \leq m$, the map

$$[0, 1] \ni t \mapsto G_m^{k, \mathbb{A}(t)}$$

is continuous and piecewise C^1 ,

(iii) for all $l, m \in \mathbb{N}$ with $m_0 \leq m \leq l$,

$$\gamma := \inf_{t \in [0, 1]} \gamma_{l, m}^{k, \mathbb{A}(t)} > 0.$$

(i) By Lemma D.3, $a := \sup_{t \in [0, 1]} a^{\mathbb{A}(t)}$ and $c := \sup_{t \in [0, 1]} c^{\mathbb{A}(t)}$ are finite. Furthermore, by Lemma D.2, there exist $0 < \lambda < 1$ and $C > 0$ such that

$$\sup_{t \in [0, 1]} \left\| \left(\widehat{\mathbb{E}}^{\mathbb{A}(t)} \right)^l \left(1 - P_{\{1\}}^{\widehat{\mathbb{E}}^{\mathbb{A}(t)}} \right) \right\| \leq C \lambda^l, \quad l \in \mathbb{N}.$$

Also, by Lemma D.2, $[0, 1] \ni t \mapsto e_{\mathbb{A}(t)}$, is continuous and we have $b := \sup_{t \in [0, 1]} \text{Tr}_{\text{Mat}_k(\mathbb{C})}(e_{\mathbb{A}(t)}) < \infty$. From these estimates and the definition of $E^{k, \mathbb{A}(t)}(N)$, $F^{k, \mathbb{A}(t)}$, we obtain the uniform bound

$$\begin{aligned} & \sup_{t \in [0, 1]} \sup_{N \geq l} \left(\sqrt{N+1} \left(3E^{k, \mathbb{A}(t)}(N)F^{k, \mathbb{A}(t)} + 2 \right) \left(E^{k, \mathbb{A}(t)}(N)F^{k, \mathbb{A}(t)} \right) + E^{k, \mathbb{A}(t)}(N) \right) \\ & \leq 4kC(kacC + c + ab)(12kC(kacC + c + ab) + 2) \sup_{N \geq l} \sqrt{N+1} \lambda^N + kacC \sup_{N \geq l} \lambda^N, \end{aligned}$$

for all $l \in \mathbb{N}$. As $0 < \lambda < 1$, the right hand side converges to 0 as $l \rightarrow \infty$. In particular, we have $l_0 := \sup_{t \in [0, 1]} \bar{l}^{k, \mathbb{A}(t)} < \infty$.

(ii) Let $\{e_{i,j}\}_{i,j=1,\dots,k}$ be the set of matrix units of $\text{Mat}_k(\mathbb{C})$. Then, for each $m \geq m_0$ and $t \in [0, 1]$, $G_m^{k, \mathbb{A}(t)}$ is the orthogonal projection onto a subspace of $\otimes_{i=0}^{m-1} \mathbb{C}^n$ spanned by the vectors $\{\Gamma_m^{k, \mathbb{A}(t)}(e_{ij}), i, j = 1, \dots, k\}$. Injectivity of $\Gamma_m^{k, \mathbb{A}(t)}$ for $m \geq m_0$ means that the dimension of $G_m^{k, \mathbb{A}(t)}$ is constant and equal to k^2 , for $t \in [0, 1]$. Hence, from Lemma E.2, (ii) holds.

(iii) For all $l, m \in \mathbb{N}$ with $m_0 \leq m \leq l$, we have $l \geq m \geq m_0 \geq m^{k, \mathbb{A}(t)}$ for $t \in [0, 1]$. Therefore, by Proposition 3.1, the lowest eigenvalue of $X(t) = (H_m^{k, \mathbb{A}(t)})_{[0, l-1]}$ is 0 and the corresponding spectral projection is $G_l^{k, \mathbb{A}(t)}$.

Therefore, the path $X : [0, 1] \mapsto X(t)$ is a continuous and piecewise C^1 -path of constant rank positive matrices. (iii) now follows from Lemma E.1 applied to this path.

Fix $m \geq m_0$ and $l > \max\{l_0, m\}$, where the max is finite by Claim (i) above. Applying Proposition 3.1, for $N \in \mathbb{N}$ with $N \geq m \geq m_0 \geq m^{k, \mathbb{A}(t)}$, the lowest eigenvalue of $(H_m^{k, \mathbb{A}(t)})_{[0, N-1]}$ is 0 and the corresponding spectral projection is $G_N^{k, \mathbb{A}(t)}$. Furthermore,

$$\frac{\gamma}{4(l+2)} \left(1 - G_N^{k, \mathbb{A}(t)} \right) \leq \frac{\gamma_{l, m}^{k, \mathbb{A}(t)}}{4(l+2)} \left(1 - G_N^{k, \mathbb{A}(t)} \right) \leq (H_m^{k, \mathbb{A}(t)})_{[0, N-1]}$$

for all $N > l$, because $\max\{\bar{l}^{k, \mathbb{A}(t)}, m\} \leq \max\{l_0, m\} < l$, where the first inequality is Claim (iii) above.

Hence, for $m \geq m_0$, the path of positive interactions $[0, 1] \ni t \mapsto \Phi_m^{k, \mathbb{A}(t)} \in \mathcal{J}_m$ is continuous and piecewise C^1 by Claim (ii), $\Phi_m^{k, \mathbb{A}(0)} = \Phi_m^{k, \mathbb{B}}$, $\Phi_m^{k, \mathbb{A}(1)} = \Phi_m^{k, \mathbb{B}'}$, and the Hamiltonian associated with $\Phi_m^{k, \mathbb{A}(t)}$ has a gap $\frac{\gamma_{l, m}^{k, \mathbb{A}(t)}}{4(l+2)}$

for each $t \in [0, 1]$, with the uniform lower bound $\inf_{t \in [0, 1]} \frac{\gamma_{l, m}^{k, \mathbb{A}(t)}}{4(l+2)} \geq \frac{\gamma}{4(l+2)} > 0$. This proves $H_m^{k, \mathbb{B}} \simeq_{C, m} H_m^{k, \mathbb{B}'}$. \square

We now turn to the problem of changing the interaction length.

Lemma 4.2. *Let $k \in \mathbb{N}$ and $\mathbb{B} \in B_{n, k}$. For any $m, m' \geq m^{k, \mathbb{B}}$,*

$$H_m^{k, \mathbb{B}} \simeq_{C, \max\{m, m'\}} H_{m'}^{k, \mathbb{B}}.$$

Proof. We may assume $m \neq m'$. For each $t \in [0, 1]$ set

$$(15) \quad \Phi(t; X) := \begin{cases} (1-t)\tau_x(1 - G_m^{k, \mathbb{B}}), & \text{if } X = [x, x+m-1] \text{ for some } x \in \mathbb{Z} \\ t\tau_x(1 - G_{m'}^{k, \mathbb{B}}), & \text{if } X = [x, x+m'-1] \text{ for some } x \in \mathbb{Z} \\ 0, & \text{otherwise} \end{cases}$$

This defines a C^1 -path $\Phi : [0, 1] \rightarrow \mathcal{J}$ such that $\Phi(0) = \Phi_m^{k, \mathbb{B}}$ and $\Phi(1) = \Phi_{m'}^{k, \mathbb{B}}$. The interaction length of $\Phi(t)$ is less than or equal to $\max\{m, m'\}$. Let $H(t)$ be the Hamiltonian associated with the interaction $\Phi(t)$. For each $N \geq \max\{m, m'\}$ and $t \in (0, 1)$, the kernel of $(H(t))_{[0, N-1]}$ is given by

$$\ker(H(t))_{[0, N-1]} = \ker(H_m^{k, \mathbb{B}})_{[0, N-1]} \cap \ker(H_{m'}^{k, \mathbb{B}})_{[0, N-1]} = \mathcal{G}_N^{k, \mathbb{B}} = \ker(H_m^{k, \mathbb{B}})_{[0, N-1]} = \ker(H_{m'}^{k, \mathbb{B}})_{[0, N-1]},$$

by Proposition 3.1. Therefore, the Hamiltonian $H(t)$ has a spectral gap, namely, for any $l > \max\{l^{k, \mathbb{B}}, m, m'\}$ and $N \geq l + 1$,

$$\frac{1}{4(l+2)} \left((1-t)\gamma_{l,m}^{k, \mathbb{B}} + t\gamma_{l,m'}^{k, \mathbb{B}} \right) (1 - G_{N,p,q}^{k, \mathbb{B}}) \leq (1-t)(H_m^{k, \mathbb{B}})_{[0, N-1]} + t(H_{m'}^{k, \mathbb{B}})_{[0, N-1]} = (H(t))_{[0, N-1]},$$

for all $t \in [0, 1]$. Hence we have $H_m^{k, \mathbb{B}} \simeq_{C, \max\{m, m'\}} H_{m'}^{k, \mathbb{B}}$. \square

Proof of Theorem 2.5. Let $k \neq k'$ and assume by contradiction that $H \simeq_C H'$. Then Theorem 2.3 yields a bijective map between ground state spaces of H and H' , in the bulk and at the edges. But Proposition 2.4(c) implies that the edge spaces of H and H' are of different dimensions, a contradiction.

Reciprocally, assume that $k = k'$. Let $H_m^{k, \mathbb{B}}, H_{m'}^{k, \mathbb{B}'} \in \mathcal{H}_k$, where $m \geq m^{k, \mathbb{B}}, m' \geq m^{k, \mathbb{B}'}$. Let $M := \max\{m, k^4 + 1\}$, $M' := \max\{m', k^4 + 1\}$, and set $\tilde{m} = \max\{M, M'\} = \max\{m, m', k^4 + 1\}$. By Lemma 4.2, we have

$$H_m^{k, \mathbb{B}} \simeq_{C, \tilde{m}} H_{\tilde{m}}^{k, \mathbb{B}}, \quad H_{m'}^{k, \mathbb{B}'} \simeq_{C, \tilde{m}} H_{\tilde{m}}^{k, \mathbb{B}'}$$

Furthermore, Lemma 4.1 yields the equivalence

$$H_{\tilde{m}}^{k, \mathbb{B}} \simeq_{C, \tilde{m}} H_{\tilde{m}}^{k, \mathbb{B}'}$$

and the theorem follows by transitivity of $\simeq_{C, \tilde{m}}$. \square

5. PIECEWISE C^1 -PATHS OF MATRICES

Recall the definitions of $\mathcal{K}_m(\mathbb{B})$ and $X_{n,k,m}$ introduced in Section 3

Proposition 5.1. *Let $n, k, m \in \mathbb{N}$ such that $2k(k-1) + 3 \leq m$. Then for any $\mathbb{A}, \mathbb{B} \in \text{Mat}_{1,n}(\text{Mat}_k(\mathbb{C}))$, there exists a continuous map $\mathbb{B} : [0, 1] \rightarrow \text{Mat}_{1,n}(\text{Mat}_k(\mathbb{C}))$, piecewise of class C^1 , such that $\mathbb{B}(0) = \mathbb{A}$, $\mathbb{B}(1) = \mathbb{B}$ and $\mathbb{B}(t) \in X_{n,k,m}$ with invertible $B_1(t)$ for $t \in (0, 1)$.*

Here, we give a constructive proof, rather than just showing the existence of the path. The strategy to prove this is to consider simple subsets of $X_{n,k,m}$ that can be constructively proven to be arcwise connected. Clearly, if $k = 1$, then $\mathcal{K}_m(\mathbb{B}) = \text{Mat}_k(\mathbb{C})$ for any $m \in \mathbb{N}$ and nonzero $\mathbb{B} \in \text{Mat}_{1,n}(\text{Mat}_k(\mathbb{C}))$. Therefore, we may assume that $2 \leq k$. Throughout the proof, we fix an orthonormal basis $\{e_\alpha\}_{\alpha=1}^k$ of $\text{Mat}_k(\mathbb{C})$.

For $k \in \mathbb{N}$, let $\mathcal{P}_k := \{(i, j) \in \{1, \dots, k\} \times \{1, \dots, k\} \mid i \neq j\}$ and

$$(16) \quad S_k := \left\{ \lambda = (\lambda_1, \dots, \lambda_k) \in (\mathbb{C} \setminus \{0\})^k \left| \begin{array}{l} \lambda_i \neq \lambda_j, \text{ if } (i, j) \in \mathcal{P}_k, \\ \frac{\lambda_i}{\lambda_j} \neq \frac{\lambda_{i'}}{\lambda_{j'}}, \text{ if } (i, j) \neq (i', j'), (i, j), (i', j') \in \mathcal{P}_k \end{array} \right. \right\}.$$

For $n, k \in \mathbb{N}$, let

$$Y_{n,k} := \left\{ \mathbb{B} \in \text{Mat}_{1,n}(\text{Mat}_k(\mathbb{C})) \left| \begin{array}{l} B_1 = \sum_{\alpha=1}^k \lambda_\alpha |e_\alpha\rangle \langle e_\alpha|, \text{ where } \lambda \in S_k, \\ \text{and } \langle B_2 e_\alpha, e_\beta \rangle \neq 0, \quad \alpha, \beta = 1, \dots, k \end{array} \right. \right\}.$$

Furthermore, for $\mathbb{B} = (B_1, \dots, B_n) \in \text{Mat}_{1,n}(\text{Mat}_k(\mathbb{C}))$ and an $R \in GL(k, \mathbb{C})$, we denote

$$R\mathbb{B}R^{-1} := (RB_1R^{-1}, \dots, RB_nR^{-1}) \in \text{Mat}_{1,n}(\text{Mat}_k(\mathbb{C})).$$

For $\mathbb{B}, \mathbb{B}' \in \text{Mat}_{1,n}(\text{Mat}_k(\mathbb{C}))$, we say that \mathbb{B} is similar to \mathbb{B}' if there exists an $R \in GL(k, \mathbb{C})$ such that $R\mathbb{B}R^{-1} = \mathbb{B}'$. Define

$$Z_{n,k} := \{\mathbb{B} \in \text{Mat}_{1,n}(\text{Mat}_k(\mathbb{C})) \mid \mathbb{B} \text{ is similar to an element in } Y_{n,k}\}.$$

Note that $R \in GL(k, \mathbb{C})$ which diagonalizes B_1 is not unique. However, for any $\mathbb{B} \in Z_{n,k}$ and invertible $P \in \text{Mat}_k(\mathbb{C})$ such that PB_1P^{-1} is diagonal with respect to the basis $\{e_\alpha\}_{\alpha=1}^k$, we have $P^{-1}\mathbb{B}P \in Y_{n,k}$. This follows from the condition $\lambda_i \neq \lambda_j$ for $(i, j) \in \mathcal{P}_k$ in the definition of S_k .

The proof of Proposition 5.1 will now proceed through a series of lemmas.

Lemma 5.2. *Let $2 \leq n, k \in \mathbb{N}$ and $2k(k-1) + 3 \leq m \in \mathbb{N}$. Then $Y_{n,k} \subset X_{n,k,m}$.*

Remark 5.3. *In particular, $X_{n,k,m}$ is nonempty for $2k(k-1) + 3 \leq m \in \mathbb{N}$. From Lemma 3.2, this means $B_{n,k}$ is not empty and contains an element \mathbb{B} with an invertible B_1 .*

Proof. Let $\mathbb{B} = (B_1, \dots, B_n) \in Y_{n,k}$. We first claim that for each $(a, b) \in \mathcal{P}_k$, there exists a nonzero vector $\zeta^{(a,b)} = (\zeta_l^{(a,b)})_{l=0, \dots, k(k-1)} \in \mathbb{C}^{k(k-1)+1}$ such that

$$(17) \quad \sum_{l=0}^{k(k-1)} \zeta_l^{(a,b)} \left(\frac{\lambda_\alpha}{\lambda_\beta} \right)^l = \delta_{\alpha,a} \delta_{\beta,b},$$

for all $\alpha, \beta = 1, \dots, k$, where $\lambda \in S_k$ are the eigenvalues of B_1 . To do this, we define for each $\alpha, \beta = 1, \dots, k$

$$v_{\alpha,\beta} := \begin{pmatrix} 1 \\ \left(\frac{\lambda_\alpha}{\lambda_\beta} \right)^1 \\ \left(\frac{\lambda_\alpha}{\lambda_\beta} \right)^2 \\ \vdots \\ \left(\frac{\lambda_\alpha}{\lambda_\beta} \right)^{k(k-1)} \end{pmatrix} \in \mathbb{C}^{k(k-1)+1}$$

The condition $\lambda \in S_k$ implies that the determinant of the following Vandermonde matrix

$$\begin{pmatrix} v_{1,1} & v_{1,2} & v_{1,3} & \cdots & v_{1,k} & v_{2,1} & v_{2,3} & \cdots & v_{2,k} & \cdots & v_{k,1} & \cdots & v_{k,k-1} \end{pmatrix} \in \text{Mat}_{k(k-1)+1}(\mathbb{C})$$

is nonzero. This means the set of vectors $\{v_{i,j}\}_{(i,j) \in \mathcal{P}_k} \cup \{v_{1,1}\}$ are linearly independent. Therefore, for each $(a, b) \in \mathcal{P}_k$, there exists a nonzero vector $\zeta^{(a,b)}$ such that

$$\zeta^{(a,b)} \perp \{v_{i,j}\}_{(i,j) \in \mathcal{P}_k, (i,j) \neq (a,b)} \cup \{v_{1,1}\},$$

and

$$\langle \zeta^{(a,b)}, v_{a,b} \rangle = 1.$$

Hence we have shown the claim.

With $\mathbb{B} \in Y_{n,k}$, equation (17) and a short calculation yield that

$$\sum_{l=0}^{k(k-1)} \zeta_l^{(a,b)} B_1^l B_2 B_1^{k(k-1)-l} = \lambda_b^{k(k-1)} \langle e_a, B_2 e_b \rangle |e_a\rangle \langle e_b|,$$

for each $(a, b) \in \mathcal{P}_k$. As $\lambda_b^{k(k-1)} \langle e_a, B_2 e_b \rangle \neq 0$, this means $|e_a\rangle \langle e_b| \in \mathcal{K}_{k(k-1)+1}(\mathbb{B})$ for any $(a, b) \in \mathcal{P}_k$.

Finally, for any (a, b) , possibly $a = b$, we choose $a', b' = 1, \dots, k$ with $a \neq a', b \neq b'$, so that $|e_a\rangle \langle e_{a'}|, |e_{b'}\rangle \langle e_b| \in \mathcal{K}_{k(k-1)+1}(\mathbb{B})$. Hence,

$$\mathcal{K}_m(\mathbb{B}) \ni |e_a\rangle \langle e_{a'}| B_2 B_1^{m-2k(k-1)-3} |e_{b'}\rangle \langle e_b| = \lambda_{b'}^{m-2k(k-1)-3} \langle e_{a'}, B_2 e_{b'} \rangle |e_a\rangle \langle e_b|.$$

As $\lambda_{b'}^{m-2k(k-1)-3} \langle e_{a'}, B_2 e_{b'} \rangle \neq 0$, this means $|e_a\rangle \langle e_b| \in \mathcal{K}_m(\mathbb{B})$ for each $a, b = 1, \dots, k$, and we conclude $\mathcal{K}_m(\mathbb{B}) = \text{Mat}_k(\mathbb{C})$. Thus we obtain $Y_{n,k} \subset X_{n,k,m}$. \square

From this, we have $Z_{n,k} \subset X_{n,k,m}$ for $2k(k-1) + 3 \leq m$. Next, we show that $Z_{n,k}$ is arcwise connected.

Lemma 5.4. *For $n, k \in \mathbb{N}$ with $n, k \geq 2$, and $\mathbb{A}, \mathbb{E} \in Z_{n,k}$, there exists a C^∞ -path $\mathbb{B} : [0, 1] \rightarrow Z_{n,k}$ such that $\mathbb{B}(0) = \mathbb{A}$, $\mathbb{B}(1) = \mathbb{E}$.*

Proof. By definition, if $\mathbb{A}, \mathbb{E} \in Z_{n,k}$, there exist $P_{\mathbb{A}}, P_{\mathbb{E}} \in GL(k, \mathbb{C})$ such that $\bar{\mathbb{A}} = (\bar{A}_1, \dots, \bar{A}_n) := P_{\mathbb{A}} \mathbb{A} P_{\mathbb{A}}^{-1} \in Y_{n,k}$ and $\bar{\mathbb{E}} = (\bar{E}_1, \dots, \bar{E}_n) := P_{\mathbb{E}} \mathbb{E} P_{\mathbb{E}}^{-1} \in Y_{n,k}$. As $GL(k, \mathbb{C})$ is connected, there exists a C^∞ -path $P : [0, 1] \rightarrow GL(k, \mathbb{C})$ such that $P(0) = P_{\mathbb{A}}$ and $P(1) = P_{\mathbb{E}}$.

By assumption, there exist $\lambda = (\lambda_1, \dots, \lambda_n), \mu = (\mu_1, \dots, \mu_n) \in S_k$ such that

$$\bar{A}_1 = \sum_{\alpha=1}^k \lambda_\alpha |e_\alpha\rangle \langle e_\alpha|, \quad \bar{E}_1 = \sum_{\alpha=1}^k \mu_\alpha |e_\alpha\rangle \langle e_\alpha|.$$

By Lemma A.3, there is a C^∞ -path $\lambda : [0, 1] \rightarrow S_k$ such that $\lambda(0) = \lambda$, and $\lambda(1) = \mu$. Furthermore, let $\xi_{\alpha,\beta} = \langle e_\alpha, \bar{A}_2 e_\beta \rangle$ and $\chi_{\alpha,\beta} = \langle e_\alpha, \bar{E}_2 e_\beta \rangle$. By assumption again, $\xi_{\alpha,\beta}, \chi_{\alpha,\beta} \neq 0$. Then Lemma A.5 yields a C^∞ -path $\xi_{\alpha,\beta} : [0, 1] \rightarrow \mathbb{C} \setminus \{0\}$ such that $\xi_{\alpha,\beta}(0) = \xi_{\alpha,\beta}$ and $\xi_{\alpha,\beta}(1) = \chi_{\alpha,\beta}$. Now, we define for $t \in [0, 1]$

$$\begin{aligned} \bar{A}_1(t) &= \sum_{\alpha=1}^k \lambda_\alpha(t) |e_\alpha\rangle \langle e_\alpha|, \\ \bar{A}_2(t) &= \sum_{\alpha,\beta=1}^k \xi_{\alpha,\beta}(t) |e_\alpha\rangle \langle e_\beta|, \\ \bar{A}_i(t) &= (1-t)\bar{A}_i + t\bar{E}_i, \quad 3 \leq i \leq n. \end{aligned}$$

Clearly, $\bar{\mathbb{A}}(t) = (\bar{A}_1(t), \dots, \bar{A}_n(t)) \in Y_{n,k}$. Finally, the path $Z_{n,k} \ni \mathbb{B}(t) = P(t)^{-1} \bar{\mathbb{A}}(t) P(t)$ is C^∞ and connects \mathbb{A} to \mathbb{E} , which concludes the proof. \square

Now we connect an arbitrary element in $\text{Mat}_{1,n}(\text{Mat}_k(\mathbb{C}))$ with an element in $Z_{n,k}$.

Lemma 5.5. *Let $n, k \in \mathbb{N}$ with $n, k \geq 2$. For any $\mathbb{A} \in \text{Mat}_{1,n}(\text{Mat}_k(\mathbb{C}))$, there exists a C^∞ -path $\mathbb{B} : [0, 1] \rightarrow \text{Mat}_{1,n}(\text{Mat}_k(\mathbb{C}))$ such that $\mathbb{B}(0) = \mathbb{A}$ and $\mathbb{B}(t) \in Z_{n,k}$ for all $t \in (0, 1]$.*

Proof. Let $\mathbb{A} = (A_1, \dots, A_n) \in \text{Mat}_{1,n}(\text{Mat}_k(\mathbb{C}))$. We consider the Jordan normal form of A_1 with respect to the orthonormal basis $\{e_\alpha\}_{\alpha=1}^k$. Let $n_1, \dots, n_M \in \mathbb{N}$ be the dimension of each Jordan cell of A_1 , so that $\sum_{l=1}^M n_l = k$. For $1 \leq l \leq M$, denote by J_l the l -th Jordan cell with eigenvalue λ_l . We further group the orthonormal basis $\{e_\alpha\}_{\alpha=1}^k$ corresponding to the decomposition, $\mathbb{C}^k = \mathbb{C}^{n_1} \oplus \dots \oplus \mathbb{C}^{n_M}$ and label them $\{f_\alpha^{(l)}\}_{\alpha=1}^{n_l}$, $l = 1, \dots, M$. For each l , $\{f_\alpha^{(l)}\}_{\alpha=1}^{n_l}$ is an orthonormal basis of \mathbb{C}^{n_l} . With these notations, each J_l can be written

$$J_l = \sum_{\alpha=1}^{n_l} \lambda_l \left| f_\alpha^{(l)} \right\rangle \left\langle f_\alpha^{(l)} \right| + \sum_{\alpha=2}^{n_l} \left| f_{\alpha-1}^{(l)} \right\rangle \left\langle f_\alpha^{(l)} \right|.$$

The Jordan normal form now reads $A_1 = R J R^{-1}$ for a $R \in GL(k, \mathbb{C})$, and where $J := J_1 \oplus \dots \oplus J_M$. It will also be useful to gather the eigenvalues with their multiplicities: for $l = 1, \dots, M$ and $\alpha = 1, \dots, n_l$, we define $\lambda_\alpha^{(l)} := \lambda_l$ and let $\lambda = (\lambda_1^{(1)}, \dots, \lambda_{n_1}^{(1)}, \lambda_1^{(2)}, \dots, \lambda_{n_{M-1}}^{(M-1)}, \lambda_1^{(M)}, \dots, \lambda_{n_M}^{(M)}) \in \mathbb{C}^k$.

For each $l = 1, \dots, M$ and $\alpha = 1, \dots, n_l$, we set $m_{(l,\alpha)} := \alpha + \sum_{i=1}^{l-1} n_i$. Let $N_{(l,\alpha)} := 2^{m_{(l,\alpha)}+1}$, and $\lambda_\alpha^{(l)}(t) := \lambda_\alpha^{(l)} + t^{N_{(l,\alpha)}}$ for $t \geq 0$. Corresponding to the decomposition $\mathbb{C}^k = \mathbb{C}^{n_1} \oplus \dots \oplus \mathbb{C}^{n_M}$, we define $\lambda(t) = (\lambda_1^{(1)}(t), \dots, \lambda_{n_1}^{(1)}(t), \lambda_1^{(2)}(t), \dots, \lambda_{n_{M-1}}^{(M-1)}(t), \lambda_1^{(M)}(t), \dots, \lambda_{n_M}^{(M)}(t)) \in \mathbb{C}^k$ and note that $\lambda(0) = \lambda$. By Lemma A.4, there exists $1 > \delta_1 > 0$ such that $\lambda(t) \in S_k$ for $t \in (0, \delta_1)$. Now, the following matrix

$$J(t) := \sum_{l=1}^M \sum_{\alpha=1}^{n_l} \lambda_\alpha^{(l)}(t) \left| f_\alpha^{(l)} \right\rangle \left\langle f_\alpha^{(l)} \right| + \sum_{l=1}^M \sum_{\alpha=2}^{n_l} \left| f_{\alpha-1}^{(l)} \right\rangle \left\langle f_\alpha^{(l)} \right|,$$

satisfies the assumptions of Lemma B.1 for each $t \in (0, \delta_1)$. Define a diagonal matrix $D(t) = D_1(t) \oplus \dots \oplus D_M(t)$ and an invertible matrix $P(t) = P_1(t) \oplus \dots \oplus P_M(t) \in GL(k, \mathbb{C})$ such that

$$D_l(t) = \sum_{\alpha=1}^{n_l} \lambda_\alpha^{(l)}(t) \left| f_\alpha^{(l)} \right\rangle \left\langle f_\alpha^{(l)} \right|, \quad P_l(t) = \sum_{\alpha,\beta=1}^{n_l} c_{\beta\alpha}^{(l)}(t) \left| f_\beta^{(l)} \right\rangle \left\langle f_\alpha^{(l)} \right|, \quad P_l(t)^{-1} = \sum_{\alpha,\beta=1}^{n_l} d_{\alpha\beta}^{(l)}(t) \left| f_\alpha^{(l)} \right\rangle \left\langle f_\beta^{(l)} \right|$$

with

$$c_{\beta\alpha}^{(l)}(t) = \begin{cases} \prod_{j=\beta}^{\alpha-1} \frac{1}{\lambda_{\alpha}^{(l)}(t) - \lambda_j^{(l)}(t)} & \alpha > \beta \\ 1 & \alpha = \beta \\ 0 & \alpha < \beta \end{cases}, \quad d_{\alpha\beta}^{(l)} = \begin{cases} \prod_{j=\alpha+1}^{\beta} \frac{1}{\lambda_{\alpha}^{(l)}(t) - \lambda_j^{(l)}(t)} & \alpha < \beta \\ 1 & \alpha = \beta \\ 0 & \alpha > \beta \end{cases}.$$

Then, by Lemma B.1, we have $J(t) = P(t)D(t)P(t)^{-1}$. As $t \mapsto \lambda(t)$ is C^∞ , $\bar{B}_1 : [0, \delta_1) \rightarrow \text{Mat}_k(\mathbb{C})$ defined by

$$\bar{B}_1(t) := RJ(t)R^{-1} = RP(t)D(t)(RP(t))^{-1}$$

is a C^∞ -path with $\bar{B}_1(0) = RJR^{-1} = A_1$.

From the representation of $P(t)$, matrix elements of $(RP(t))^{-1}A_2RP(t)$ are of the form

$$(18) \quad \left\langle f_{\beta}^{(l)}, P(t)^{-1} (R^{-1}A_2R) P(t) f_{\beta'}^{(l')} \right\rangle = \left\langle f_{n_l}^{(l)}, (R^{-1}A_2R) f_1^{(l')} \right\rangle d_{\beta n_l}^{(l)}(t) c_{1\beta'}^{(l')}(t) + g_{\beta\beta'}^{(l')}(t),$$

where

$$(19) \quad \left| g_{\beta\beta'}^{(l')}(t) \right| \leq \|R^{-1}A_2R\| \cdot \sum_{\substack{\alpha \geq \beta, \alpha' \leq \beta', \\ (\alpha, \alpha') \neq (n_l, 1)}} \left| \frac{d_{\beta\alpha}^{(l)}(t)}{d_{\beta n_l}^{(l)}(t)} \right| \left| \frac{c_{\alpha'\beta'}^{(l')}(t)}{c_{1\beta'}^{(l')}(t)} \right| \left| d_{\beta n_l}^{(l)}(t) c_{1\beta'}^{(l')}(t) \right| \leq \|R^{-1}A_2R\| \cdot \left| d_{\beta n_l}^{(l)}(t) c_{1\beta'}^{(l')}(t) \right| \cdot k^2 t^2,$$

for $t \in (0, \delta_1)$. Here we used the estimate

$$\begin{aligned} \left| \frac{d_{\beta\alpha}^{(l)}(t)}{d_{\beta n_l}^{(l)}(t)} \right| &= \prod_{j=\alpha+1}^{n_l} \left| \lambda_{\beta}^{(l)}(t) - \lambda_j^{(l)}(t) \right| = \prod_{j=\alpha+1}^{n_l} |t^{N_{(l,\beta)}} - t^{N_{(l,j)}}| \leq t^2, \\ \left| \frac{c_{\alpha'\beta'}^{(l')}(t)}{c_{1\beta'}^{(l')}(t)} \right| &= \prod_{j=1}^{\alpha'-1} \left| \lambda_{\beta'}^{(l')}(t) - \lambda_j^{(l')}(t) \right| = \prod_{j=1}^{\alpha'-1} |t^{N_{(l',\beta')}} - t^{N_{(l',j')}}| \leq t^2, \end{aligned}$$

if $\beta \leq \alpha \leq n_l - 1$, $2 \leq \alpha' \leq \beta'$, for $t \in (0, \delta_1) \subset (0, 1)$.

Now we define $\bar{B}_2(t)$ for each $t \in [0, \delta_1)$ as

$$\bar{B}_2(t) = A_2 + tR \left(\sum_{l=1}^M \sum_{l'=1}^M \left| f_{n_l}^{(l)} \right\rangle \left\langle f_1^{(l')} \right| \right) R^{-1}.$$

This gives a C^∞ -path in $\text{Mat}_k(\mathbb{C})$ with $\bar{B}_2(0) = A_2$.

We claim that there exists $\delta_1 \geq \delta_2 > 0$ such that the matrix elements of $(RP(t))^{-1}\bar{B}_2(t)RP(t)$ are nonzero for $t \in (0, \delta_2)$. A simple computation using (18) yields the bound

$$\left| \left\langle f_{\beta}^{(l)}, (RP(t))^{-1}\bar{B}_2(t)RP(t) f_{\beta'}^{(l')} \right\rangle \right| \geq \left| \left\langle f_{n_l}^{(l)}, (R^{-1}A_2R) f_1^{(l')} \right\rangle + t \left| d_{\beta n_l}^{(l)}(t) c_{1\beta'}^{(l')}(t) - g_{\beta\beta'}^{(l')}(t) \right| \right|.$$

where $d_{\beta n_l}^{(l)}(t) c_{1\beta'}^{(l')}(t) \neq 0$ by the definition for $t \in (0, \delta_1)$. Since there is $0 < \delta_2 < \delta_1$ such that

$$\|R^{-1}A_2R\| \cdot k^2 t^2 < |\langle f_{n_l}^{(l)}, (R^{-1}A_2R) f_1^{(l')} \rangle + t|, \quad t \in (0, \delta_2)$$

the bound (19) implies the claim.

Finally, we choose $0 < \delta < \delta_2$ and define a path $\mathbb{B} : [0, 1] \rightarrow \text{Mat}_{1,n}(\text{Mat}_k(\mathbb{C}))$ by $B_1(t) := \bar{B}_1(\delta t)$, $B_2(t) := \bar{B}_2(\delta t)$, and $B_i(t) := A_i$ for $i = 3, \dots, n$. By the above, $(RP(t))^{-1}\mathbb{B}(t)(RP(t)) \in Y_{n,k}$ for all $t \in (0, 1]$. Hence $\mathbb{B}(t) \in Z_{n,k}$. Furthermore, \mathbb{B} is C^∞ and $\mathbb{B}(0) = \mathbb{A}$, and we have obtained a path satisfying the conditions in the Lemma. \square

With this, we are now ready to prove the main proposition of this section.

Proof of Proposition 5.1. Recall that it suffices to consider the case $k \geq 2$. For any $\mathbb{A}, \mathbb{E} \in \text{Mat}_{1,n}(\text{Mat}_k(\mathbb{C}))$, Lemma 5.5 yields two C^∞ -paths $\mathbb{B}_{\mathbb{A}}, \mathbb{B}_{\mathbb{E}} : [0, 1] \rightarrow \text{Mat}_{1,n}(\text{Mat}_k(\mathbb{C}))$ such that $\mathbb{B}_{\mathbb{A}}(0) = \mathbb{A}$, $\mathbb{B}_{\mathbb{E}}(0) = \mathbb{E}$ and $\mathbb{B}_{\mathbb{A}}(t), \mathbb{B}_{\mathbb{E}}(t) \in Z_{n,k} \subset X_{n,k,m}$ for all $t \in (0, 1]$. By Lemma 5.4, $\mathbb{B}_{\mathbb{A}}(1), \mathbb{B}_{\mathbb{E}}(1) \in Z_{n,k}$ can be connected by a

C^∞ -path $\mathbb{B}_{\text{mid}} : [0, 1] \rightarrow \text{Mat}_{1,n}(\text{Mat}_k(\mathbb{C}))$ in $Z_{n,k} \subset X_{n,k,m}$. Hence, the path $\mathbb{B} : [0, 1] \rightarrow \text{Mat}_{1,n}(\text{Mat}_k(\mathbb{C}))$ defined by

$$\mathbb{B}(t) := \begin{cases} \mathbb{B}_{\mathbb{A}}(3t) & 0 \leq t \leq 1/3 \\ \mathbb{B}_{\text{mid}}(3(t - 1/3)) & 1/3 < t \leq 2/3 \\ \mathbb{B}_{\mathbb{E}}(3(1 - t)) & 2/3 \leq t \leq 1 \end{cases}$$

is a continuous path with $\mathbb{B}(0) = \mathbb{A}$ and $\mathbb{B}(1) = \mathbb{E}$ and such that $\mathbb{B}(t) \in Z_{n,k} \subset X_{n,k,m}$ for $t \in (0, 1)$. It is everywhere continuously differentiable but at $t = 1/3$ and $t = 2/3$. Furthermore $\mathbb{B}(t) \in Z_{n,k}$ implies the invertibility of $B_1(t)$ for $t \in (0, 1)$. \square

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APPENDIX A. CONTINUOUS PATHS IN S_k

We denote the Euclidean norm of \mathbb{R}^k by $\|\cdot\|_{\mathbb{R}^k}$.

Lemma A.1. *Let $k, l \in \mathbb{N}$ with $l + 1 < k$ and N an l -dimensional submanifold of \mathbb{R}^k without boundary. Suppose that $\xi : [0, 1] \rightarrow \mathbb{R}^k$ is a C^∞ -path, such that $\xi(0), \xi(1) \in N^c$. Then for any $\varepsilon > 0$, there exists a C^∞ -path $\xi_\varepsilon : [0, 1] \rightarrow \mathbb{R}^k$ such that*

$$(20) \quad \begin{aligned} & \sup_{t \in [0, 1]} \|\xi_\varepsilon(t) - \xi(t)\|_{\mathbb{R}^k} < \varepsilon, \\ & \xi_\varepsilon([0, 1]) \cap N = \emptyset, \\ & \xi_\varepsilon(0) = \xi(0), \quad \xi_\varepsilon(1) = \xi(1). \end{aligned}$$

Proof. Let $\delta = \min\{\frac{\varepsilon}{2}, d_{\mathbb{R}^k}(\xi(0), N), d_{\mathbb{R}^k}(\xi(1), N)\} > 0$. Define a closed subset C and an open subset U in $[0, 1]$ by

$$\begin{aligned} C &:= \{t \in [0, 1] \mid d_{\mathbb{R}^k}(\xi(t), N) \leq \frac{\delta}{3}\}, \\ U &:= \{t \in [0, 1] \mid d_{\mathbb{R}^k}(\xi(t), N) < \frac{2\delta}{3}\}. \end{aligned}$$

Then C is a subset of U and there exists a C^∞ -path $\gamma : [0, 1] \rightarrow [0, 1]$ such that $\gamma|_C = 1$ and $\gamma|_{U^c} = 0$.

Let S be an open ball in \mathbb{R}^k centered at the origin with radius δ . Define a C^∞ -map $F : [0, 1] \times S \rightarrow \mathbb{R}^k$ by

$$F(t, s) := \xi(t) + \gamma(t)s.$$

The map F is transversal to N . For if $(t, s) \in F^{-1}(N)$, then $\gamma(t) \neq 0$, by the definition of γ . For any $t \in [0, 1]$ with $\gamma(t) \neq 0$, the map $S \rightarrow \mathbb{R}^k$ given by $s \mapsto \xi(t) + \gamma(t)s$ is a submersion. Hence, F is a submersion at (t, s) with $\gamma(t) \neq 0$, in particular, at $(t, s) \in F^{-1}(N)$. Therefore, F is transversal to N .

For the restriction ∂F of F to the boundary $\{0, 1\} \times S$, $(\partial F)^{-1}(N) = \emptyset$. Hence ∂F is transversal to N .

We conclude by the Transversality Theorem (see e.g. [GP]) that for almost every $s \in S$, the map $\xi_s(t) := \xi(t) + \gamma(t)s$ is transversal to N . Since $\dim(N) + 1 = l + 1 < k$, transversality implies that N and ξ_s do not intersect. Furthermore, $\xi_s : [0, 1] \rightarrow \mathbb{R}^k$ is C^∞ . Since $\|\xi_s(t) - \xi(t)\| \leq s < \varepsilon$ and $\gamma(0) = \gamma(1) = 0$, this concludes the proof. \square

Lemma A.2. *Let $m, k \in \mathbb{N}$ and N_1, \dots, N_m be submanifolds of \mathbb{R}^k without boundary with $\dim N_i + 1 < k$, $i = 1, \dots, m$. Suppose that $\xi : [0, 1] \rightarrow \mathbb{R}^k$ is a C^∞ -map with $\xi(0), \xi(1) \in \cap_{i=1}^m N_i^c$. Then, there exists a C^∞ -map $\zeta : [0, 1] \rightarrow \mathbb{R}^k$ such that*

$$(21) \quad \begin{aligned} \zeta([0, 1]) \cap \left(\bigcup_{i=1}^m N_i \right) &= \emptyset, \\ \xi(0) &= \zeta(0), \quad \xi(1) = \zeta(1). \end{aligned}$$

Proof. We consider the following statement for $j = 1, \dots, m$.

(P_j): There exists a C^∞ -map $\xi_j : [0, 1] \rightarrow \mathbb{R}^k$ such that

$$(22) \quad \begin{aligned} \xi_j([0, 1]) \cap \left(\bigcup_{i=1}^j N_i \right) &= \emptyset, \\ \xi(0) &= \xi_j(0), \quad \xi(1) = \xi_j(1), \\ \sup_{t \in [0, 1]} \|\xi(t) - \xi_j(t)\|_{\mathbb{R}^k} &< \left(1 - \frac{1}{2^j}\right). \end{aligned}$$

We prove (P_m) inductively, and set $\zeta = \xi_m$. The statement (P₁) is obtained by applying Lemma A.1 to N_1 , ξ , and $\varepsilon = \frac{1}{2} > 0$. Assume that (P_j) is true for $j < m$. Applying Lemma A.1 to N_{j+1} , ξ_j and

$$\varepsilon_{j+1} = \min \left\{ 2^{-j-1}, d_{\mathbb{R}^k}(\xi_j([0, 1], \bigcup_{i=1}^j N_i)) \right\} > 0,$$

we obtain a C^∞ -path $\xi_{j+1} : [0, 1] \rightarrow \mathbb{R}^k$ satisfying

$$(23) \quad \begin{aligned} \xi_{j+1}([0, 1]) \cap N_{j+1} &= \emptyset, \\ \xi(0) &= \xi_j(0) = \xi_{j+1}(0), \quad \xi(1) = \xi_j(1) = \xi_{j+1}(1), \\ \sup_{t \in [0, 1]} \|\xi_{j+1}(t) - \xi_j(t)\|_{\mathbb{R}^k} &< \varepsilon_{j+1}. \end{aligned}$$

By the choice of $\varepsilon_{j+1} \leq d_{\mathbb{R}^k}(\xi_j([0, 1], \bigcup_{i=1}^j N_i))$ and the last inequality in (23), we conclude that ξ_{j+1} does not intersect with $\bigcup_{i=1}^j N_i$. Again, by the choice of $\varepsilon_j \leq 2^{-j-1}$ and the last inequality in (23) we have $\sup_{t \in [0, 1]} \|\xi(t) - \xi_{j+1}(t)\|_{\mathbb{R}^k} < (1 - \frac{1}{2^{j+1}})$. \square

Lemma A.3. *Let S_k be the subset in \mathbb{C}^k defined by (16). Then for any points $\lambda, \mu \in S_k$, there exists a C^∞ -path $\zeta : [0, 1] \rightarrow \mathbb{C}^k$ such that $\zeta([0, 1]) \subset S_k$ and $\zeta(0) = \lambda$, $\zeta(1) = \mu$.*

Proof. We identify \mathbb{C}^l with \mathbb{R}^{2l} for any $l \in \mathbb{N}$ naturally. For $j = 1, \dots, k$, and $(i, j) \in \mathcal{P}_k$, set

$$M_j := \{v \in \mathbb{C}^k \mid v_j = 0\}, \quad Z_{(i,j)} := \{v \in \mathbb{C}^k \mid v_i = v_j\}.$$

Furthermore, for any $(i, j), (l, m) \in \mathcal{P}_k$ with $(i, j) \neq (l, m)$, set

$$N_{(i,j),(l,m)} := \{v \in \mathbb{C}^k \mid v_i v_m = v_j v_l\} \cap \left(\bigcap_{j=1}^k M_j^c \right).$$

Clearly M_j , $j = 1, \dots, k$ and $Z_{(i,j)}$, $(i, j) \in \mathcal{P}_k$ are $2k - 2$ dimensional submanifolds of \mathbb{R}^{2k} without boundary. For any $(i, j), (l, m) \in \mathcal{P}_k$ with $(i, j) \neq (l, m)$, the map $\mathbb{C}^k \ni v \rightarrow v_i v_m - v_j v_l \in \mathbb{C}$ is a submersion on $\bigcap_{j=1}^k M_j^c$. Therefore, $N_{(i,j),(l,m)}$ is a $2k - 2$ dimensional submanifold of \mathbb{R}^{2k} without boundary. It is easy to see

$$S_k = \left(\bigcap_{j=1}^k M_j^c \right) \cap \left(\bigcap_{(i,j) \in \mathcal{P}_k} Z_{(i,j)}^c \right) \cap \left(\bigcap_{(i,j),(l,m) \in \mathcal{P}_k, (i,j) \neq (l,m)} N_{(i,j),(l,m)}^c \right).$$

Let $\xi : [0, 1] \rightarrow \mathbb{C}^k$ be a C^∞ -path defined by

$$\xi(t) = (1 - t)\lambda + t\mu, \quad t \in [0, 1],$$

connecting λ and μ . Applying Lemma A.2 to ξ and the finite set of $2k - 2$ dimensional submanifolds M_j , $Z_{(i,j)}$, $N_{(i,j),(l,m)}$ of \mathbb{R}^{2k} without boundary, we obtain the result. \square

Lemma A.4. *Let $\lambda = (\lambda_1, \dots, \lambda_k) \in \mathbb{C}^k$, and $n_1, \dots, n_k \in \mathbb{N}$ with $n_i \neq n_j$ if $(i, j) \in \mathcal{P}_k$. Let $\lambda(t) = (\lambda_1(t), \dots, \lambda_k(t)) \in \mathbb{C}^k$ be defined by*

$$\lambda_i(t) := \left(\lambda_i + t^{2^{n_i}} \right),$$

for $t \in \mathbb{R}$. Then there exists $\delta > 0$ such that $\lambda(t) \in S_k$ for all $t \in (0, \delta)$.

Proof. First we show that for $n, m, l, k \in \mathbb{N}$,

$$2^n + 2^m = 2^l + 2^k,$$

implies either $n = l$ and $m = k$, or $n = k$ and $m = l$. To see this, we may assume $n \geq m, l \geq k$. We have

$$2^m(2^{n-m} + 1) = 2^k(2^{l-k} + 1).$$

If $n \neq m$ and $k \neq l$, then $m = k$. This implies $n - m = l - k$ that $n = l$. If $n = m$, then we have $2^{m+1} = 2^k(2^{l-k} + 1)$. Therefore, $l = k$ and $k = m = n$. Similarly, if $l = k$, then $n = m = k = l$.

Let $\lambda_1, \dots, \lambda_k \in \mathbb{C}$, and $n_1, \dots, n_k \in \mathbb{N}$ with $n_i \neq n_j$ if $(i, j) \in \mathcal{P}_k$.

For each $i = 1, \dots, k$, by the analiticity of polynomial, there exists a $\delta_i > 0$ such that

$$\lambda_i + t^{2^{n_i}} \neq 0,$$

for all $t \in (0, \delta_i)$. For any $(i, j) \in \mathcal{P}_k$, there exists a $\delta_{ij} > 0$ such that

$$\lambda_i + t^{2^{n_i}} \neq \lambda_j + t^{2^{n_j}}$$

for all $t \in (0, \delta_{ij})$, as $n_i \neq n_j$. For any $(i, j), (i', j') \in \mathcal{P}_k$ with $(i, j) \neq (i', j')$, from the above claim, we have

$$2^{n_i} + 2^{n_{j'}} \neq 2^{n_j} + 2^{n_{i'}}.$$

Therefore, the polynomial

$$\left(\lambda_i + t^{2^{n_i}} \right) \left(\lambda_{j'} + t^{2^{n_{j'}}} \right) - \left(\lambda_j + t^{2^{n_j}} \right) \left(\lambda_{i'} + t^{2^{n_{i'}}} \right)$$

is not zero. Hence there exists a $\delta_{(ij), (i'j')} > 0$ such that

$$\left(\lambda_i + t^{2^{n_i}} \right) \left(\lambda_{j'} + t^{2^{n_{j'}}} \right) \neq \left(\lambda_j + t^{2^{n_j}} \right) \left(\lambda_{i'} + t^{2^{n_{i'}}} \right)$$

for $t \in (0, \delta_{(ij), (i'j')})$. Hence setting $\delta := \min\{\delta_i, \delta_{i,j}, \delta_{(i,j), (i',j')}\}$, we have $\lambda(t) \in S_k$ for all $t \in (0, \delta)$. \square

We close this section with a simple lemma.

Lemma A.5. *Let F be a finite subset of \mathbb{C} and let $\chi, \eta \in \mathbb{C}$ with $\chi \neq \eta$. Then there exists a C^∞ -map $\xi : [0, 1] \rightarrow \mathbb{C}$ with $\xi(0) = \chi$, $\xi(1) = \eta$ and $\xi(t) \in F^c$ for all $t \in (0, 1)$, such that*

$$|\xi(t) - \chi| \leq 2|\eta - \chi|, \quad t \in [0, 1].$$

Proof. This can be done by a modification of the path $[0, 1] \ni t \mapsto (1-t)\chi + t\eta$ avoiding F . \square

APPENDIX B. PERTURBATION OF JORDAN MATRICES

Here, we consider matrices $A \in \text{Mat}_k(\mathbb{C})$ that are close to a Jordan matrix in the sense that A has the same block form as a Jordan matrix, but in each block, all diagonal elements are different. We exhibit explicitly the matrix diagonalizing it.

Lemma B.1. *Let $k \in \mathbb{N}$ and $n_1, \dots, n_M \in \mathbb{N}$ such that $n_1 + \dots + n_M = k$. Decompose $\mathbb{C}^k = \mathbb{C}^{n_1} \oplus \dots \oplus \mathbb{C}^{n_M}$ and let $\{f_\alpha^{(l)}\}_{\alpha=1}^{n_l}$ be an orthonormal basis of \mathbb{C}^{n_l} , for each $l = 1, \dots, M$. Let $\{\lambda_\alpha^{(l)}\}_{\alpha=1, \dots, n_l, l=1, \dots, M}$ be distinct elements in \mathbb{C} . Define*

$$J_l := \sum_{\alpha=1}^{n_l} \lambda_\alpha^{(l)} \left| f_\alpha^{(l)} \right\rangle \left\langle f_\alpha^{(l)} \right| + \sum_{\alpha=2}^{n_l} \left| f_{\alpha-1}^{(l)} \right\rangle \left\langle f_\alpha^{(l)} \right|,$$

for $l = 1, \dots, M$, and let $J := J_1 \oplus \dots \oplus J_M$. Then there exists a diagonal matrix D and an invertible matrix P such that

$$J = PDP^{-1}.$$

Here $D = D_1 \oplus \dots \oplus D_M$, $P = P_1 \oplus \dots \oplus P_M$ with respect to the decomposition $\mathbb{C}^k = \mathbb{C}^{n_1} \oplus \dots \oplus \mathbb{C}^{n_M}$, and

$$D_l = \sum_{\alpha=1}^{n_l} \lambda_{\alpha}^{(l)} |f_{\alpha}^{(l)}\rangle \langle f_{\alpha}^{(l)}|, \quad P_l = \sum_{\alpha, \beta=1}^{n_l} c_{\beta\alpha}^{(l)} |f_{\beta}^{(l)}\rangle \langle f_{\alpha}^{(l)}|, \quad P_l^{-1} = \sum_{\alpha, \beta=1}^{n_l} d_{\alpha\beta}^{(l)} |f_{\alpha}^{(l)}\rangle \langle f_{\beta}^{(l)}|$$

with

$$c_{\beta\alpha}^{(l)} = \begin{cases} \prod_{j=\beta}^{\alpha-1} \frac{1}{\lambda_{\alpha}^{(l)} - \lambda_j^{(l)}} & \alpha > \beta \\ 1 & \alpha = \beta \\ 0 & \alpha < \beta \end{cases}, \quad d_{\alpha\beta}^{(l)} = \begin{cases} \prod_{j=\alpha+1}^{\beta} \frac{1}{\lambda_{\alpha}^{(l)} - \lambda_j^{(l)}} & \alpha < \beta \\ 1 & \alpha = \beta \\ 0 & \alpha > \beta \end{cases}.$$

Proof. This follows by the matrix diagonalizations of the blocks J_l . □

APPENDIX C. PRIMITIVE MAPS

In this section we collect known results about positive maps on $\text{Mat}_k(\mathbb{C})$. We refer the reader to the literature for proofs of the stated theorems, as for example to the notes [W], and references therein.

Theorem C.1. *Let $T : \text{Mat}_k(\mathbb{C}) \rightarrow \text{Mat}_k(\mathbb{C})$ be a positive linear map. The following properties are equivalent:*

- (i) *There is no nontrivial orthogonal projection P such that $T(P \text{Mat}_k(\mathbb{C}) P) \subset P \text{Mat}_k(\mathbb{C}) P$,*
- (ii) *For any nonzero $A \geq 0$ and $t > 0$, $\exp(tT)(A) > 0$*

Remark C.2. *A positive map satisfying the above equivalent conditions is said to be irreducible.*

We say that λ is a nondegenerate eigenvalue of T if the corresponding projection $P_{\{\lambda\}}^T$ is one dimensional. Irreducible positive maps satisfy the following properties.

Theorem C.3. *Let $T : \text{Mat}_k(\mathbb{C}) \rightarrow \text{Mat}_k(\mathbb{C})$ be a nonzero irreducible positive linear map. Then the spectral radius r_T of T is a strictly positive, non-degenerate eigenvalue with a strictly positive eigenvector h_T :*

$$T(h_T) = r_T h_T > 0.$$

Theorem C.4. *Let $T : \text{Mat}_k(\mathbb{C}) \rightarrow \text{Mat}_k(\mathbb{C})$ be a unital completely positive map and let*

$$T(A) = \sum_{i=1}^n B_i A B_i^*$$

be its Kraus decomposition. Let $\mathbb{B} := (B_1, \dots, B_n)$. Then the following properties are equivalent:

- (i) *There exists $l \in \mathbb{N}$ such that $T^l(A) > 0$ for any nonzero $A \geq 0$,*
- (ii) *There exists a unique faithful T -invariant state φ , and it satisfies*

$$\lim_{l \rightarrow \infty} T^l(A) = \varphi(A)1, \quad A \in \text{Mat}_k(\mathbb{C}),$$

- (iii) *$\sigma(T) \cap \{z \in \mathbb{C} : |z| \geq 1\} = \{1\}$, 1 is a nondegenerate eigenvalue of T , and there exists a faithful T -invariant state,*
- (iv) *There exists $m \in \mathbb{N}$ such that $\mathcal{K}_m(\mathbb{B}) = \text{Mat}_k(\mathbb{C})$, where $\mathcal{K}_m(\mathbb{B})$ was defined in (6),*
- (v) *There exists $m \in \mathbb{N}$ such that $\mathcal{K}_l(\mathbb{B}) = \text{Mat}_k(\mathbb{C})$, for all $l \geq m$.*

Remark C.5. *A unital completely positive map satisfying the above (equivalent) conditions is said to be primitive.*

APPENDIX D. CONTINUOUS PATHS OF POSITIVE MAPS

We gather simple results for spectral quantities of elements in \mathcal{T}_k or \mathcal{T}_k , see Section 2, and on continuous paths of such maps.

Lemma D.1. *For any $T \in \mathcal{T}_k$,*

$$e_T = P_{\{r_T\}}^T(1)$$

is a nonzero element in $\text{Mat}_k(\mathbb{C})_+$. Moreover, there exists a unique $r_T^{-1}T$ -invariant state φ_T on $\text{Mat}_k(\mathbb{C})$ such that for any $a \in \text{Mat}_k(\mathbb{C})$,

$$P_{\{r_T\}}^T(a) = \varphi_T(a)e_T,$$

and $\varphi_T(e_T) = 1$.

Proof. For $T \in \mathcal{T}_k$, note that $Te_T = r_T e_T$. By the definition of \mathcal{T}_k , we see that

$$(24) \quad P_{\{r_T\}}^T(a) = \lim_{n \rightarrow \infty} r_T^{-n} T^n(a),$$

for all $a \in \text{Mat}_k(\mathbb{C})$. From this and the positivity of T , $e_T = P_{\{r_T\}}^T(1) \geq 0$. Again by the positivity of T and (24), $e_T = 0$ would imply $P_{\{r_T\}}^T = 0$, which is not true. Therefore, e_T is nonzero. Furthermore, as the range $P_{\{r_T\}}^T$ is the one dimensional ray of e_T , there exists a linear functional φ_T such that

$$(25) \quad P_{\{r_T\}}^T(a) = \varphi_T(a)e_T,$$

for all $a \in \text{Mat}_k(\mathbb{C})$. Finally, from (24), φ_T is a $r_T^{-1}T$ -invariant state. As $\varphi_T(e_T)e_T = P_{\{r_T\}}^T(e_T) = e_T$, we have $\varphi_T(e_T) = 1$. \square

Lemma D.2. *For a continuous and piecewise C^1 -path $T : [0, 1] \rightarrow \mathcal{T}_k$, the corresponding paths e_{T_t} , φ_{T_t} , and r_{T_t} are continuous and piecewise C^1 . Moreover, there exist $0 < \lambda < 1$ and $c > 0$ such that*

$$\sup_{t \in [0, 1]} \left\| r_{T_t}^{-l} T_t^l (1 - P_{\{r_{T_t}\}}^{T_t}) \right\| \leq c \lambda^l,$$

for all $l \in \mathbb{N}$.

Proof. For each $t_0 \in [0, 1]$, $T_{t_0} \in \mathcal{T}_k$. Therefore, there exists a δ_{t_0} with $\frac{1}{3}r_{T_{t_0}} > \delta_{t_0} > 0$ such that $\sigma(T_{t_0} \setminus \{r_{T_{t_0}}\}) \subset B_{(r_{T_{t_0}} - 3\delta_{t_0})}(0)$. Fix such a δ_{t_0} . Then we have

$$(26) \quad P_{\{r_{T_{t_0}}\}}^{T_{t_0}} = \frac{1}{2\pi i} \oint_{|z - r_{T_{t_0}}| = \delta_{t_0}} (z - T_{t_0})^{-1} dz.$$

By the continuity of T , there exists an $\varepsilon_{t_0} > 0$ such that

$$(27) \quad \sigma(T_t) \subset (\sigma(T_{t_0}))_{\frac{\delta_{t_0}}{2}}$$

for any $t \in (t_0 - \varepsilon_{t_0}, t_0 + \varepsilon_{t_0}) \cap [0, 1]$. For this $\varepsilon_{t_0} > 0$ fixed,

$$(t_0 - \varepsilon_{t_0}, t_0 + \varepsilon_{t_0}) \cap [0, 1] \ni t \mapsto Q_t := \frac{1}{2\pi i} \oint_{|z - r_{T_{t_0}}| = \delta_{t_0}} (z - T_t)^{-1} dz$$

is well-defined, continuous and piecewise C^1 . From the continuity of the path of projections Q_t , and the fact that $P_{\{r_{T_{t_0}}\}}^{T_{t_0}} = Q_{t_0}$ is one dimensional, we see that each Q_t is a one dimensional projection. Hence Q_t is a one dimensional projection corresponding to the spectrum $\sigma(T_t) \cap B_{\delta_{t_0}}(r_{T_{t_0}})$. From this, (27) and the fact $r_{T_t} \in \sigma(T_t)$, we get

$$(28) \quad \{r_{T_t}\} = \sigma(T_t) \cap B_{\delta_{t_0}}(r_{T_{t_0}}).$$

Therefore we obtain

$$Q_t = P_{\{r_{T_t}\}}^{T_t}, \quad t \in (t_0 - \varepsilon_{t_0}, t_0 + \varepsilon_{t_0}) \cap [0, 1].$$

In particular, $[0, 1] \ni t \mapsto P_{\{r_{T_t}\}}^{T_t}$ is continuous and piecewise C^1 .

From this and the following formulae,

$$(29) \quad e_{T_t} = P_{\{r_{T_t}\}}^{T_t}(1), \quad \varphi_{T_t} = \frac{\text{Tr}_{\text{Mat}_k(\mathbb{C})} \left(P_{\{r_{T_t}\}}^{T_t}(\cdot) \right)}{\text{Tr}_{\text{Mat}_k(\mathbb{C})}(e_{T_t})}, \quad r_{T_t} = \frac{\text{Tr}_{\text{Mat}_k(\mathbb{C})}(T_t(e_{T_t}))}{\text{Tr}_{\text{Mat}_k(\mathbb{C})}(e_{T_t})},$$

we see that e_{T_t} , φ_{T_t} , and r_{T_t} are continuous and piecewise C^1 .

To prove the second part, for each $t_0 \in [0, 1]$, we take $\delta_{t_0} > 0$ and $\varepsilon_{t_0} > 0$ as above. From (27) and (28), we have

$$\sigma(T_t) \subset B_{(r_{T_{t_0}} - \frac{\varepsilon}{2}\delta_{t_0})}(0) \cup \{r_{T_t}\}, \quad t \in (t_0 - \varepsilon_{t_0}, t_0 + \varepsilon_{t_0}) \cap [0, 1].$$

By the continuity of r_{T_t} , this implies the existence of $0 < \varepsilon'_{t_0} < \varepsilon_{t_0}$ and $0 < \delta'_{t_0} < 1$ such that

$$\sigma(r_{T_t}^{-1}T_t) \setminus \{1\} \subset B_{1-\delta'_{t_0}}(0), \quad t \in (t_0 - \varepsilon'_{t_0}, t_0 + \varepsilon'_{t_0}) \cap [0, 1].$$

By the compactness of $[0, 1]$, there exist finite number of $t_1, \dots, t_m \in [0, 1]$ such that

$$[0, 1] = \cup_{i=1}^m (t_i - \varepsilon'_{t_i}, t_i + \varepsilon'_{t_i}) \cap [0, 1].$$

Set $\delta := \min\{\delta'_i \mid i = 1, \dots, m\} > 0$. Then we have

$$\sigma(r_{T_t}^{-1}T_t) \setminus \{1\} \subset B_{1-\delta}(0), \quad t \in [0, 1].$$

Setting

$$c := \sup_{(z,t): |z|=1-\delta, t \in [0,1]} \|(z - r_{T_t}^{-1}T_t)^{-1}\| < \infty,$$

we obtain

$$\|r_{T_t}^{-l}T_t^l(1 - P_{\{r_{T_t}\}}^{T_t})\| = \left\| \frac{1}{2\pi i} \oint_{|z|=1-\delta} z^l (z - r_{T_t}^{-1}T_t)^{-1} dz \right\| \leq c(1-\delta)^l, \quad l \in \mathbb{N},$$

and the claim follows with $\lambda = 1 - \delta$. \square

By definition, the two quantities

$$a_T = \|(pe_T p)^{-1}\|_{\text{Mat}_k(\mathbb{C})}, \quad c_T = \|(q\rho_T q)^{-1}\|_{\text{Mat}_k(\mathbb{C})},$$

are finite.

Lemma D.3. *Let $k \in \mathbb{N}$ and p, q be two fixed orthogonal projections on $\text{Mat}_k(\mathbb{C})$. For any continuous path $T : [0, 1] \rightarrow \mathcal{T}_k$, $a_{T_t, p}$ and $c_{T_t, q}$ are continuous. In particular, $\sup_{t \in [0, 1]} a_{T_t, p} < \infty$, and $\sup_{t \in [0, 1]} c_{T_t, q} < \infty$.*

Proof. By the proof of the previous lemma and since $\mathcal{T}_k \subset \mathcal{T}_k$, e_{T_t} and φ_{T_t} are continuous and so is ρ_{T_t} . Hence, $t \mapsto a_{T_t, p}$ and $t \mapsto c_{T_t, q}$ are continuous as well, and since they are defined on a compact set, they are uniformly bounded. \square

APPENDIX E. PATH OF VECTOR SPACES

The proof of the following lemma is standard.

Lemma E.1. *Let $k, m \in \mathbb{N}$ with $k \leq m$. Let $X : [0, 1] \rightarrow (\text{Mat}_m(\mathbb{C}))_+$ be continuous and piecewise C^1 -path of positive matrices such that the rank of $X(t)$ is k for all $t \in [0, 1]$. Let $S(t)$ be the support projection of $X(t)$, and set $\gamma(t) := d_{\mathbb{C}}(\sigma(X(t)) \setminus \{0\}, \{0\})$. Then, the path of projections*

$$S : [0, 1] \ni t \mapsto S(t) \in \text{Mat}_m(\mathbb{C})$$

is continuous and piecewise C^1 and

$$\inf_{t \in [0, 1]} \gamma(t) > 0.$$

Lemma E.2. *Let $l, k, m \in \mathbb{N}$ with $k \leq m$. Let $\psi_i : [0, 1] \rightarrow \mathbb{C}^m$, $i = 1, \dots, l$ be continuous and piecewise C^1 -paths of vectors in \mathbb{C}^m , such that*

$$\dim \text{span}\{\psi_i(t)\}_{i=1}^l = k, \quad t \in [0, 1].$$

For each $t \in [0, 1]$, let $S(t)$ be orthogonal projection onto the span of $\{\psi_i(t)\}_{i=1}^l$. Then, the path of projections

$$S : [0, 1] \ni t \mapsto S(t) \in \text{Mat}_m(\mathbb{C})$$

is continuous and piecewise C^1 .

Proof. Define

$$X(t) := \sum_{i=1}^l |\psi_i(t)\rangle \langle \psi_i(t)|.$$

Then $X : [0, 1] \ni t \mapsto X(t) \in \text{Mat}_m(\mathbb{C})_+$ defines a continuous and piecewise C^1 -path, and $S(t)$ is the support projection of $X(t)$. Hence the rank of $X(t)$ is k for all $t \in [0, 1]$. Applying Lemma E.1, we obtain the claim. \square

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